On Active Disturbance Rejection Control for Nonlinear Systems Using Time-Varying Gain*

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Abstract

In this paper, we propose a modified nonlinear extended state observer (ESO) with a time-varying gain in active disturbance rejection control (ADRC) to deal with a class of nonlinear systems which are essentially normal forms of general affine nonlinear systems. The total disturbance which includes unknown dynamics of the system, external disturbance, and unknown part of the control coefficient is estimated through ESO and is canceled in nonlinear feedback loop. The practical stability for the resulting closed-loop is obtained. It is shown that the “peaking value” occurred often in the constant high gain design can be significantly reduced by the time-varying gain approach.

\textbf{Key words.} Disturbance rejection, extended state observer, nonlinear systems.

\textbf{AMS subject classifications.} 93C10, 93C15, 34D20.

1 Introduction

In the past three decades, many control approaches have been developed to cope with system uncertainty or external disturbances. Basically, there are two types of strategies to deal with uncertainty. One focus on the worst case scenario which makes the controller designed conservative. This can be found in the sliding mode control and the high gain control, among many others. The other is first to estimate uncertainty and then cancel the effect of uncertainty in feedback loop. The

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latter idea can be found typically in adaptive control, the internal model principle, and external principle ([21]). However, in adaptive control, the updated parameters are not always convergent, and the internal and external model principle requires priority knowledge of dynamics of unknown disturbance.

The active disturbance rejection control (ADRC), as an unconventional design strategy similar to the external model principle ([21]), was first proposed by Han in 1998 based on realistic rethinking about the PID technology that has dominated the control engineering for almost one century ([13]). The uncertainties dealt with by ADRC are much more complicated. For instance, ADRC can deal with the coupling between the external disturbances, the system un-modeled dynamics, and the superadded unknown part of control input. The most remarkable feature of ADRC is that the disturbance is estimated, in real time, through an extended state observer and is canceled in the feedback loop. This reduces significantly the control energy in practice ([32]).

In the past two decades, ADRC has been successfully applied to many engineering control problems as reviewed in hysteresis compensation ([7]), high pointing accuracy and rotation speed ([20]), noncircular machining ([26]), fault diagnosis ([27]), high-performance motion control ([23]), chemical processes ([28]), vibrational MEMS gyroscopes ([29, 31]), tension and velocity regulations in Web processing lines ([14]), DC-DC power converter ([24]), among many others. In all applications in process control and motion control, compared with the huge literature of control theory in dealing with uncertainty such as system un-modeled dynamics ([5]), external disturbance rejection ([4]), and unknown parameters ([3]), the ADRC has exhibited remarkable characteristics of independent of mathematical modes like PID-control; whether it is on high accuracy control of micron grade or integrated control of very large scale. It is now generally acknowledged that the ADRC is a new control strategy that is capable of dealing with un-modeled dynamics and external disturbance, regardless of nonlinearity, time-variance in systems. For instance, by the internal model principle for output regulation, it requires the availability of disturbance dynamics whereas by ADRC, only upper bound of external disturbance is needed.

The design of ADRC can be split into three steps. The first step is to design a tracking differentiator (TD). Since in many practical applications, we know only reference signal \( v \) itself, while in feedback, we need the derivatives of \( v \). This gives rise to differential tracking problem. Han initially proposed a noise-tolerance TD to recover the derivatives of the reference signal \( v \) ([11]). For general treatment on differential tracking, we refer to [19] and the references therein. The second step towards ADRC is to design an “extended state observer” (ESO) to estimate not only the state of system but also the total disturbance. The last step is the ESO-based feedback control ([10]), which is somehow the separation principle in nonlinear system control.

The convergence of linear/or nonlinear TD is reported in [8, 10, 12]. The convergence of linear ESO is available in [30], and very recently, nonlinear ESO has been studied in [10] where linear ESO is a special case of the nonlinear ones. It is shown that with an appropriate choice of nonlinear functions in ESO such as weighted homogeneous functions can improve the accuracy and reduce the peaking value, with the same constant high gain ([10]). The convergence of linear ADRC, which
is based on linear ESO and linear feedback, is investigated in [15] and the convergence of nonlinear ADRC has been proven in [9]. In all these works, no matter linear or nonlinear, the ESO uses constant high gain tuning parameter. However, the constant high gain tuning parameter causes notorious “peaking value problem” in the initial time stage. In [6], a saturated function method is applied to reduce the peaking value but the bound of initial values is assumed preliminarily.

In this paper, we consider the following nonlinear system:

\[
\begin{cases}
\dot{x}(t) = A_n x(t) + B_n [f(t, x(t), \zeta(t), w(t)) + b(t)u(t)], \\
\dot{\zeta}(t) = f_0(t, x(t), \zeta(t), w(t)), \\
y(t) = C_n x(t),
\end{cases}
\tag{1.1}
\]

where \( x \in \mathbb{R}^n \) and \( \zeta \in \mathbb{R}^m \) are the system states, \( w \in \mathbb{C}^1([0, \infty), \mathbb{R}) \) is the external disturbance, \( A_n, B_n \) are defined as 

\[
A_n = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \quad B_n^T = C_n = (0, 0, \ldots, 1), \tag{1.2}
\]

\( f \in \mathbb{C}(\mathbb{R}^{n+m+2}, \mathbb{R}), f_0 \in \mathbb{C}(\mathbb{R}^{n+m+2}, \mathbb{R}^m) \) are unknown nonlinear functions, \( u(t) \in \mathbb{R} \) is the input (control), and \( y(t) = C_n x(t) = x_1(t) \) is the output (measurement). The control coefficient \( b(t) \), with nominal value \( b_0 \neq 0 \), contains some uncertainty.

System (1.1) is quite general. It is actually the normal form of \((n + m)\)-order affine nonlinear systems with relative degree \( n \). According to [16], if a general \((n + m)\)-order affine nonlinear system

\[
\begin{cases}
\dot{x} = \phi(x) + \varphi(x)u, \\
y = h(x)
\end{cases}
\]

has a relative degree \( n \), then it can be transformed into the form

\[
\begin{cases}
\dot{x} = A_n x + B_n [f(x, \zeta) + b(x, \zeta)u], \\
\dot{\zeta} = f_0(x, \zeta), \\
y = C_n x.
\end{cases}
\]

System (1.1) may also arise in models of mechanical and electromechanical systems. Examples can be found in [20, 24, 26, 32, 31, 28].

We design a nonlinear ESO with time-varying gain for system (1.1) as follows:

\[
\begin{cases}
\dot{x}_1(t) = \dot{x}_2(t) + \frac{1}{r^{n-1}(t)} g_1(r^n(t) \delta(t)), \\
\dot{x}_2(t) = \dot{x}_3(t) + \frac{1}{r^{n-2}(t)} g_2(r^n(t) \delta(t)), \\
\quad \vdots \\
\dot{x}_n(t) = \dot{x}_{n+1}(t) + gn(r^n(t) \delta(t)) + b_0 u(t), \\
\dot{x}_{n+1}(t) = r(t)g_{n+1}(r^n(t) \delta(t)),
\end{cases}
\tag{1.3}
\]

where \( \delta(t) = x_1(t) - y(t), r(t) \) is the time-varying gain to be increased gradually. When \( r(t) \equiv 1/\varepsilon \), (1.3) is reduced to the constant high gain nonlinear ESO in [10]. The aim of ESO is to estimate the states \( x_1, x_2, \ldots x_n \) and total disturbance

\[
x_{n+1}(t) \triangleq f(t, x(t), \zeta(t), w(t)) + [b(t) - b_0]u(t), \tag{1.4}
\]
which is also called the extended state.

The high gain observer has been studied extensively. A recent work is reviewed in [18]. Observer with time-varying gain (updated gain or dynamic gain) is also used in [1, 2, 22], where the gain is a dynamics determined by some nonlinear function related to control plant. The choice of our time-varying gain is flexible. Basic requirement is that the time-varying gain should grow from a small value to maximal value to reduce the peaking value. The major difference between [1, 2] and this paper is that there is no estimation for uncertainty in these works. Only in [22], a constant unknown nominal control value is estimated on stabilization for an affine nonlinear system. The estimation/cancelation nature of ADRC makes it very different.

The main contribution of this paper is that we introduce a type of time varying-gain in observer (1.3) to achieve peaking value reduction observed by the constant high gain in [10]. In addition, the nonlinear functions in (1.1) can be Hölder continuous rather than Lipschitz continuous assumed in our previous works [10, 9].

We proceed as follows. In Section 2, we give the main results of ESO (1.3) based feedback control. Some numerical simulations are presented for illustrations. The proof for the main results is presented in Section 3.

2 Main results

Let us first recall the whole process of ADRC for system (1.1). The first part of ADRC is tracking differentiator (TD). For a reference signal $v$, we use the following TD to estimate its derivatives ([11]):

$$
\begin{align*}
\dot{z}_1(t) &= z_2(t), \\
\vdots & \\
\dot{z}_n(t) &= z_{n+1}(t), \\
\dot{z}_{n+1}(t) &= -\rho^{n+1}k_1(z_1(t) - v(t)) - \rho^n k_2 z_2(t) - \cdots - \rho k_{n+1} z_{n+1}(t).
\end{align*}
$$

By theorem 3.1 of [11], if $\sup_{t \in [0,\infty)} |v^{(i)}(t)| < \infty$, $i = 1, 2, \ldots, n$, and the following matrix is Hurwitz

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-k_1 & -k_2 & -k_3 & \cdots & -k_n
\end{pmatrix}
$$

then for any $a > 0$, $|z_i(t) - v^{(i-1)}(t)| \leq \bar{M}/\rho$, $i = 1, 2, \ldots, n + 1$ uniformly in $t \in [a, \infty)$, where $\bar{M}$ is a $\rho$-independent constant. It is noted that if the derivatives of $v$ are available, we just let $z_i = v^{(i-1)}$.

Since the TD part is relatively independent of other two parts of ADRC, we do not couple TD in the closed loop; instead, we use $z_i$ directly in the feedback loop.
The second part of ADRC is the ESO (1.3) that estimates both state and the total disturbance of system (1.1).

Suppose that we have obtained estimates for both state and total disturbance. We then use estimation/cancelation strategy to design the ESO-based output feedback control as follows:

\[ u = \frac{1}{b_0}(u_0(x_1 - z_1, \ldots, x_n - z_n) + z_{n+1} - \hat{x}_{n+1}), \]

where \( \hat{x}_{n+1} \) is used to compensate the total disturbance \( x_{n+1} \) and \( u_0 \) is the nominal control to be specified later. The objective of the control is to make the error \( (x_1 - z_1, x_2 - z_2, \ldots, x_n - z_n) \) be convergent to zero as time goes to infinity in the prescribed way. Precisely,

\[ x_i(t) - z_i(t) = y^{(i-1)}(t) \text{ for } i = 1, 2, \ldots, n \text{ as } t \to \infty, \]

where \( y \) satisfies

\[ y^{(n)}(t) = u_0 \left( y(t), y'(t), \ldots, y^{(n-1)}(t) \right), \]

and \( u_0 \) is chosen so that (2.7) is asymptotically stable. It is worth pointing out that, unlike to the high gain dominate approach in dealing with uncertainty (see, e.g., ([25, Lemmas 2.2 and 2.4], [1]), we do not need the high gain in the feedback loop.

The ADRC can handle at least two types of control problems: system stabilization (when \( v \equiv 0 \)) and output regulation (when \( v \neq 0 \)).

To prove the convergence, we need some assumptions. The following Assumption is about the unknown system functions \( f \) and \( f_0 \), the control coefficient \( b(t) \), and the external disturbance \( w(t) \).

**Assumption A1.** There exist positive constants \( M_0, K_0, K_1, K_2 \), and function \( \varpi \in C(\mathbb{R}, \mathbb{R}^+) \) such that

(I) \( \sup_{t \in [0, \infty)} \{ |w(t)|, |\dot{w}(t)|, |b(t)|, |\dot{b}(t)| \} \leq M_0 \);

(II) \( \sum_{i=1}^{m} \left| \frac{\partial f(t, x, \zeta, w)}{\partial \zeta_i} \right| + \sum_{i=1}^{n} \left| \frac{\partial f(t, x, \zeta, w)}{\partial x_i} \right| \leq K_0 + \varpi(w), t \in [0, \infty), x \in \mathbb{R}^n, \zeta \in \mathbb{R}^m, w \in \mathbb{R}; \)

(III) \( \left| \frac{\partial f}{\partial t}(t, x, \zeta, w) \right| + \left| \frac{\partial f}{\partial w}(t, x, \zeta, w) \right| + |f(t, x, \zeta, w)| + \|f_0(t, x, \zeta, w)\| \leq K_1 + K_2|x| + \varpi(w). \)

From Assumption A1, we can see that the conditions on system functions and external disturbance are quite general. However, in practice, Assumption A1 cannot be verified beforehand due to involving of unknown functions. In this regard, the “test” and “try” strategy should be performed in applications to see if the unknown functions are in our class.

From the second inequality in (I) of Assumption A1 and Theorem 3.2 of [11], for any \( \sigma > 0 \) and \( a > 0 \), there exists \( R > 0 \) such that \( |z_i(t) - y^{(i-1)}(t)| \leq \sigma \) and \( \|z_1(t), \ldots, z_{n+1}(t), \hat{z}_{n+1}(t)\| \leq M_1 \) uniformly on \( [a, \infty) \), where \( M_1 > M_0 \) is an \( R \)-dependent constant.

For the nonlinear functions \( g_i \)’s in ESO (1.3), we need the following Assumption.

**Assumption A2.** The nonlinear function \( g_i : \mathbb{R} \to \mathbb{R} \) is Hölder continuous and satisfies the following Lyapunov conditions: There exist radially unbounded, positive definite functions \( V \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^+), W \in C(\mathbb{R}^{n+1}, \mathbb{R}^+) \) such that for any \( t \in \mathbb{R}^{n+1}, \)

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Theorem 2.1. Suppose that Assumptions A1, A2, A3, A4 are satisfied, \(b_0 > 0\), \(\sup_{t \in [0, \infty)} |b(t) - b_0|/b_0| \leq \min\{1/2, 1/N_0\}\), and the gain function in ESO (1.3) satisfies (2.10). Then the closed-loop composed of system (1.1), ESO (1.3), and output feedback (2.6) has the following practical stability.

Essentially, the key step towards the existence of the Lyapunov functions \(V, W\) in assumption A2 is that \(g_i\)'s should be chosen so that zero equilibrium of the following system is asymptotically stable

\[
\begin{aligned}
\dot{\xi}_1 &= \xi_1 - g_1(\xi), \\
&\vdots \\
\dot{\xi}_n &= \xi_{n+1} - g_n(\xi), \\
\dot{\xi}_{n+1} &= -g_{n+1}(\xi).
\end{aligned}
\]  

In [10], there are two ways to construct \(g_i\). One is linear function and other is weighted homogeneous function. Concrete examples are given in Example 2.1.

The next Assumption A3 below is about function \(u_0(\cdot)\) in feedback control (2.6).

**Assumption A3.** All partial derivatives of \(u_0(\cdot)\) are globally bounded by \(L\), and there exist continuous, radially unbounded positive definite functions \(V \in C^1(\mathbb{R}^n, \mathbb{R}^+), W \in C(\mathbb{R}^n, \mathbb{R}^+)\) such that for any \(\tau \in \mathbb{R}^n, \)

- \(\sum_{i=1}^{n-1} \tau_i \frac{\partial V}{\partial \tau_i}(\tau) + u_0(\tau) \frac{\partial V}{\partial \tau_n}(\tau) \leq -W(\tau), \|\tau\|^2 + \left| \frac{\partial V}{\partial \tau_n}(\tau) \right|^2 \leq N_2 W(\tau), N_2 > 0.\)

Since \(V, W, V, W\) are continuous, radially unbounded, and positive definite functions in Assumptions A2 and A3, it follows from lemma 4.3 of [17, p.145] that there exist continuous class \(K_0\) functions \(\kappa_i, \xi_i (i = 1, 2, 3, 4)\) such that for any \(t \in \mathbb{R}^n, \tau \in \mathbb{R}^n, \)

\[
\begin{align}
\kappa_1(\|\xi\|) &\leq V(\xi) \leq \kappa_2(\|\xi\|), \\
\kappa_3(\|\tau\|) &\leq W(\tau) \leq \kappa_4(\|\tau\|), \\
\xi_1(\|\tau\|) &\leq V(\tau) \leq \xi_2(\|\tau\|), \\
\xi_3(\|\tau\|) &\leq W(\tau) \leq \xi_4(\|\tau\|).
\end{align}
\]  

Assumption on these class \(K_0\) functions is given below.

**Assumption A4.** There exists an \(N_3 > 0\) such that \(\kappa_4(s) \leq N_3 \xi_3(s)\) for any \(s \geq 0\).

The gain function \(r(t) \in C(\mathbb{R}^+, \mathbb{R}^+)\) is initialized from a small value and then increase continuously to a large number:

\[
r(t) = \begin{cases} 
  e^{at}, & 0 \leq t < \frac{1}{a} \ln r_0, \\
  r_0, & t \geq \frac{1}{a} \ln r_0,
\end{cases}
\]  

where \(r_0\) is a large number so that the errors between the solutions of (1.3) and (1.1) are in the prescribed scale, \(a > 0\) is a constant to control the increasing speed of \(r\).
convergence results: for any given \( \sigma > 0 \), there exists a constant \( r^* > 0 \) (the relation with \( \sigma \) can be seen in (3.66)) such that for any \( r_0 > r^* \),

\[
|x_i(t) - \hat{x}_i(t)| < \sigma, \ |x_j(t) - z_j(t)| < \sigma, \ t > t_0,
\]

where \( t_0 \) is an \( r_0 \)-dependent constant, \( i = 1, 2, \ldots, n + 1 \).

**Remark 2.1.** If we do not consider the total disturbance estimation and peaking value problem, the practical stability can also be achieved by the high-gain dominate control method in [1].

If in “total disturbance”, there is something known, the ESO should make use of this information as much as possible. In what follows, we discuss the case where \( f \) satisfies

\[
f(t, x, \zeta, w) = \tilde{f}(x) + \bar{f}(t, \zeta, w),
\]

and \( \tilde{f} \) in (2.11) is known. In this case, the ESO for (1.1) is modified as follows:

\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_2(t) + \frac{1}{r_{n-1}(t)} g_1(r^n(t) \delta(t)), \\
\vdots \\
\dot{x}_n(t) &= \dot{x}_{n+1}(t) + g_n(r^n(t) \delta(t)) + \tilde{f}(\dot{x}_1(t), \ldots, \dot{x}_n(t)) + b_0 u(t), \\
\dot{x}_{n+1}(t) &= r(t) g_{n+1}(r^n(t) \delta(t)),
\end{align*}
\]

where the total disturbance in this case becomes

\[
x_{n+1}(t) = \tilde{f}(t, \zeta(t), w(t)) + (b(t) - b_0) u(t).
\]

To deal with this problem, we need some conditions directly about \( \tilde{f}, \bar{f}, f_0 \).

**Assumption A1.**

- \( \tilde{f} \) is Hölder continuous, that is,

\[
|\tilde{f}(\tau) - \tilde{f}(\hat{\tau})| \leq L \|\tau - \hat{\tau}\|^\beta, \ \forall \ \tau, \hat{\tau} \in \mathbb{R}^n, \beta > 0;
\]

- There exist constant \( K > 0 \) and function \( \varpi_1 \in C(\mathbb{R}, \mathbb{R}^+) \) such that

\[
\sum_{i=1}^m \left| \frac{\partial \tilde{f}}{\partial \zeta_i}(t, \zeta, w) \right| + \left| \frac{\partial \tilde{f}}{\partial w}(t, \zeta, w) \right| + \left| \frac{\partial \tilde{f}}{\partial t}(t, \zeta, w) \right| + \|f_0(t, x, \zeta, w)\| \leq K + \varpi_1(w),
\]

for all \( (t, x, \zeta, w) \in \mathbb{R}^{n+m+2} \).

The feedback control in this case is changed accordingly into

\[
u(t) = \frac{1}{b_0}(u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t)) - \tilde{f}(\dot{x}_1(t), \ldots, \dot{x}_n(t)) + z_{n+1}(t) - \hat{x}_{n+1}(t)),
\]

where \( u_0 \in C^1(\mathbb{R}^n, \mathbb{R}) \) is a nonlinear function satisfying Assumption A3.
Theorem 2.2. Suppose that Assumptions A1 and A3 are satisfied, \( \sup_{t \in [0, \infty)} \max \{|w(t)|, |\dot{w}(t)|\} < \infty \), \( b(t) \equiv b_0 \), and \( g_i \)'s satisfy (I) of Assumption A2 with

\[
\left| \frac{\partial V}{\partial r_{n+1}} (t) \right| + \| \epsilon \|^\beta \left| \frac{\partial V}{\partial r_{n+1}} (t) \right| \leq NW(t), \quad \forall \| \epsilon \| \geq R, \quad R > 0. \tag{2.16}
\]

Then the closed loop of system (1.1) with ESO (2.12) and the feedback control (2.15) has the following practical convergence results: for any given \( \sigma > 0 \), \( i = 1, 2, \ldots, n + 1 \), \( j = 1, 2, \ldots, n \), there exists an \( r^* > 0 \) such that for all \( r_0 > r^* \),

\[
|x_i(t) - \hat{x}_i(t)| < \sigma, \quad |x_j(t) - z_j(t)| < \sigma, \quad t > t_0,
\]

where \( t_0 \) is an \( r_0 \)-dependent constant.

To end this section, we present some numerical simulations for illustration.

Example 2.1. Consider the following control system:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= f(t, x(t), \zeta(t), w(t)) + b(t)u(t), \\
\dot{\zeta}(t) &= f_0(t, x_1(t), x_2(t), \zeta(t), w(t)), \\
y(t) &= x_1(t),
\end{align*}
\tag{2.17}
\]

where \( f \) and \( f_0 \) are unknown system functions, \( w \) is the unknown external disturbance, control parameter \( b \) is also unknown but its nominal value \( b_0 \) is given, and \( y \) is the output. The control purpose is to design the output feedback control so that \( y \) tracks reference signal \( v \).

In the numerical simulation, we set

\[
\begin{align*}
f_0(t, x_1, x_2, \zeta, w) &= a_1(t)x_1 + \sin x_2 + \cos \zeta + w, \\
f(t, x_1, x_2, \zeta, w) &= a_2(t)x_2 + \sin(x_1 + x_2) + \zeta + w, \\
a_1(t) &= 1 + \sin t, \quad a_2(t) = 1 + \cos t, \\
b(t) &= 10 + 0.1 \sin t, \quad b_0 = 10, \quad w(t) = 1 + \cos t + \sin 2t, \quad v(t) = \sin t,
\end{align*}
\tag{2.18}
\]

It is easily to verify that Assumption A1 is satisfied.

To approximate the states \( x_1, x_2 \) and the extended state

\[
x_3(t) \triangleq f(t, x_1(t), x_2(t), \zeta(t), w(t)) + (b(t) - b_0)u(t)
\tag{2.19}
\]

of system (2.1) by output \( y = x_1 \), the ESO is designed as (1.3) with \( n = 2 \). The functions \( g_1, g_2 \) and \( g_3 \) are chosen in the spirit of Assumption A2. Let us give some cues for choosing these functions. Similar with [10], the most simple functions are linear ones: \( g_i(\tau) = k_\tau, \tau \in \mathbb{R} \) and \( k_i \)'s are chosen such that the following matrix is Hurwitz

\[
K = \begin{pmatrix}
-k_1 & 1 & 0 \\
-k_2 & 0 & 1 \\
-k_3 & 0 & 0
\end{pmatrix}.
\tag{2.20}
\]
It is readily to prove that the above linear functions \( g_i \) adding with a Lipschitz continuous nonlinear function with small Lipschitz constant also satisfy Assumption A2. Let \( g_1(s) = 6s + \phi(s), \ g_2(s) = 11s, \ g_3(s) = 6s, \ s \in \mathbb{R} \), where \( \phi \in \mathbb{C}(\mathbb{R}, \mathbb{R}) \) is a Lipschitz continuous nonlinear function satisfying \( s\phi(s) < 0 \) given by

\[
\phi(s) = \begin{cases} 
-\frac{1}{4\pi}, & s \in (-\infty, -\pi/2), \\
\frac{\sin s}{\pi}, & s \in (-\pi/2, \pi/2), \\
\frac{1}{4\pi}, & s \in (\pi/2, \infty).
\end{cases}
\]  

(2.21)

The Lyapunov functions \( \mathcal{V} \) and \( \mathcal{W} \) in Assumption A2 can be defined as

\[
\mathcal{V}(\tau_1, \tau_2, \tau_3) = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix} Q \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix}^\top, \quad \mathcal{W}(\tau) = \frac{1}{2} \begin{pmatrix} \tau_1^2 + \tau_2^2 + \tau_3^2 \end{pmatrix},
\]

where \( Q \) is the positive definite matrix solution to the following Lyapunov equation:

\[
K^\top Q + QK = -I_{3\times3},
\]

with \( I_{3\times3} \) being the \( 3 \times 3 \) identity matrix and \( K \) given in (2.20). A direct computation shows that

\[
\mathcal{V}(\tau) = 1.7\tau_1^2 + 0.7\tau_2^2 + 1.5333\tau_3^2 - \tau_1\tau_2 - 1.4\tau_1\tau_3 - \tau_2\tau_3,
\]

and the class \( \mathcal{K}_\infty \) functions in Assumption A2 are now

\[
\kappa_1(s) = 0.09s^2, \quad \kappa_2(s) = 2.33s^2, \quad \kappa_3(s) = \kappa_4(s) = 0.5s^2, \quad s \geq 0.
\]

We can then verify that these \( g_i \)'s satisfy Assumption A2.

The gain parameter is chosen as (2.10) with \( a = 5, \ r_0 = 200 \). For notation simplicity, we use directly \( z_1(t) = \sin t, \ z_2(t) = \cos t, \ z_3(t) = -\sin t \) as the target states. In the feedback control (2.6), the nonlinear function \( u_0 \) can be chosen as a linear function adding with a Lipschitz continuous function \( \phi \) defined in (2.21):

\[
u_0(\tau_1, \tau_2) = -2\tau_1 - 4\tau_2 - \phi(\tau_1).
\]

(2.22)

The Lyapunov functions \( V \) and \( W \) can be defined as \( V(\tau_1, \tau_2) = (\tau_1, \tau_2)P(\tau_1, \tau_2)^\top \) and \( W(\tau_1, \tau_2) = 0.5\tau_1^2 + 0.5\tau_2^2 \), where \( P \) is the positive definite matrix solution to the following Lyapunov equation:

\[
\begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix}^\top P + P \begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix} = -I_{2\times2}.
\]

A direct computation shows that

\[
V(\tau_1, \tau_2) = 1.375\tau_1^2 + 0.1875\tau_2^2 + 0.5\tau_1\tau_2.
\]

The \( \mathcal{K}_\infty \) functions \( \kappa_i, \ i = 1, 2, 3, 4 \) in Assumption A3 are given as

\[
\kappa_1(s) = 0.13s^2, \quad \kappa_2(s) = 1.43s^2, \quad \kappa_3(s) = \kappa_4(s) = 0.5s^2, \quad s \geq 0.
\]

(2.23)

We can then readily check that Assumptions A3 and A4 are satisfied.
The numerical results are plotted in Figure 1. From Figure 1, we can see that: a) \( \hat{x}_i (i = 1, 2, 3) \) of (1.3) converge to \( x_1, x_2 \) of system (2.1) and its total disturbance given in (2.19) in a relative short time. The most remarkable fact is that peaking disappears; b) Under the feedback control, the output \( x_1 \) and its derivative \( x_2 \) track reference signal \( \sin t \) and its derivative \( \cos t \) satisfactorily. However the price of peaking value reduction by the time-varying gain ESO is slightly longer for tracking.

To compare the time-varying ESO over constant high gain ESO in [10], we give numerical simulations for system (2.17) by constant high gain ESO which is a special case of (1.3) by setting \( r(t) \equiv r_0 \). The tracking time of constant ESO can be prolonged as the gain constant \( r_0 \) becomes small. In order taking relatively long time as possibly long as in Figure 1, we choose a much smaller \( r_0 = 20 \). Let the other functions and parameters are the same as that in Figure 1. The results are plotted in Figure 2. It is seen that the constant gain ESO can rapidly track the states \( x_1, x_2 \) and total disturbance \( x_3 \). The convergence time in this case is also short than time-varying gain ESO in Figure 1. We see from Figure 2(b) and 2(c) that very large peaking values in \( \hat{x}_2 \) around 14 and \( \hat{x}_3 \) around 150 are observed at the initial stage in Figure 2. But contrast sharply with Figure 2, in Figure 1, the peaking values for \( \hat{x}_2 \) and \( \hat{x}_3 \) are complete negligible.

At the end of this section, we use a simple example to explain that the peaking value of ESO by the constant high gain occurs only at the initial stage and the time-varying gain ESO has much smaller peaking value.

Finally, if in (2.17),

\[
f(t, x_1, x_2, \zeta, w) = \tilde{f}(x_1, x_2) + \bar{f}(t, \zeta, w),
\]
\[
w(t) = 1 + \cos t + \sin 2t, \quad \tilde{f}(t, \zeta, w) = \cos(t + \zeta) + w, \\
\bar{f}(x_1, x_2) = [x_1]^\beta \triangleq \text{sgn}(x_1)|x_1|^\beta, \quad 0 < \beta < 1, 
\] (2.24)

then we can easily check that \( f \) does not satisfy Assumption A1, but satisfies Assumption A1\(^*\).

Suppose that \( \tilde{f} \) is known. The total disturbance is
\[
x_3(t) \triangleq \bar{f}(t, \zeta(t), w(t)). 
\] (2.25)

Then the ESO can be designed as (2.12) with
\[
g_1(\nu) = [\nu]^{\theta}, \quad g_2(\nu) = [\nu]^{2\theta-1}, \quad g_3(\nu) = [\nu]^{3\theta-2}, \quad \nu \in \mathbb{R}, \ \theta = 0.8, \ \beta = 0.1,
\]

be chosen such that the following vector field composed by \( g_1, g_2, g_3 \)
\[
G(t_1, t_2, t_3) = \begin{pmatrix}
\nu_2 - k_1g_1(t_1) \\
\nu_3 - k_2g_2(t_1) \\
-k_3g_3(t_1)
\end{pmatrix}
\]
is weighted homogeneous [10]. With the argument in [10, p. 425], we can prove that \( g_i \)'s satisfy Assumption A2\(^{*\ast}\). The gain parameter is the same as that in (2.10) with \( a = 5, \ r_0 = 200. \) The feedback control is designed as (2.15) with \( u_0(x_1, x_2) = -2x_1 - 4x_2 - \phi(x_1) - [\hat{x}_1]^\beta. \) We can verify that all conditions of Theorem 2.2 are satisfied. The numerical results are plotted in Figure 3.

From Figure 3, we see that the states and the total disturbance are convergent very satisfactorily

\[\text{(a) } x_1 \text{ (green), } \hat{x}_1 \text{ (red), } \sin t \text{ (blue) } \quad \text{(b) } x_2 \text{ (green), } \hat{x}_2 \text{ (red), } \cos t \text{ (blue) } \quad \text{(c) } x_3 \text{ (green), } \hat{x}_3 \text{ (red)}\]

Figure 3: Numerical results of (2.17) for \( f \) to be chosen as (2.24).

and there is no peaking value problem. Meanwhile, the system state \( x_1 \) and its derivative \( x_2 \) track very well the reference signal \( \sin t \) and its derivative \( \cos t \).

To end this section, we use a simple example to explain why the time-varying gain ESO can reduce the peaking value dramatically. The considered system is the following second order system with external disturbance:
\[
\begin{cases}
\dot{x}_1(t) = x_2(t), \\
\dot{x}_2(t) = u(t) + d(t), \\
y = x_1,
\end{cases}
\] (2.26)

where \( u \) is the control input, \( d \in C^1([0, \infty), \mathbb{R}) \) is the external disturbance with \( \sup_{t \in [0, \infty)} |\dot{d}(t)| \leq M, \ M > 0. \) For the sake of simplicity, we just use the following LESO to estimate state and
external disturbance:
\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_2(t) + 3r(t)(x_1(t) - \hat{x}_1(t)), \\
\dot{x}_2(t) &= \dot{x}_3(t) + 3r^2(t)(x_1(t) - \hat{x}_1(t)), \\
\dot{x}_3(t) &= r^3(t)(x_1(t) - \hat{x}_1(t)),
\end{align*}
\tag{2.27}
\]
It is easy to verify the following matrix \( E \) is Hurwitz
\[
E = \begin{pmatrix}
-3 & 1 & 0 \\
-3 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}.
\tag{2.28}
\]
Hence, we can verify that Assumption A2 is satisfied with \( V(\nu) = \nu P \nu^\top \), where \( P \) is the positive definite matrix solution of the Lyapunov equation \( EP + PE^\top = -I_3 \) with \( I_3 \) the three dimensional identity matrix. Let
\[
\eta_i = r^{3-i}(t)(x_i - \hat{x}_i), \quad i = 1, 2, 3, \quad \eta(t) = (\eta_1(t), \eta_2(t), \eta_3(t))^\top.
\tag{2.29}
\]
Then
\[
\dot{\eta}(t) = -r(t)E\eta(t) + \left( \frac{2\dot{r}(t)}{r(t)}, \frac{\dot{r}(t)}{r(t)}, \dot{d}(t) \right)^\top.
\tag{2.30}
\]
If \( r(t) \equiv r_0, \ r_0 > 0 \), then a straightforward computation shows that
\[
\begin{pmatrix}
\eta_1(t) \\
\eta_2(t) \\
\eta_3(t)
\end{pmatrix}
= e^{-r_0Et} \begin{pmatrix}
r_0^2(x_1(0) - \hat{x}_1(0)) \\
r_0(x_2(0) - \hat{x}_2(0)) \\
x_3(0) - \hat{x}_3(0)
\end{pmatrix}
+ \int_0^t e^{-r_0E(t-s)} \begin{pmatrix}
0 \\
0 \\
\dot{d}(s)
\end{pmatrix} ds.
\tag{2.31}
\]
By the definition of \( \eta_i \),
\[
\begin{align*}
\dot{x}_2 &= \frac{1}{r_0} \sum_{j=1}^3 d_{2j} e^{r_0\lambda_j} r_0^{3-j} (x_j(0) - \hat{x}_j(0)), \\
\dot{x}_3 &= \sum_{j=1}^3 d_{3j} e^{r_0\lambda_j} r_0^{3-j} (x_j(0) - \hat{x}_j(0)),
\end{align*}
\tag{2.32}
\]
where \( \lambda_i < 0 \) \((i = 1, 2, 3)\) are the eigenvalues of \( E \), \( d_{ij} (i, j = 1, 2, 3) \) are reals determined by the matrix \( E \). It is seen that the peaking value occurs only near the initial stage since for any \( a > 0 \), \( \eta(t) \to 0 \) as \( r \to \infty \) uniformly in \( t \in [a, \infty) \). On the other hand, in the initial time stage however, \( e^{r_0\lambda_i} \) is very close to 1. This is the reason behind for the peaking value problem by the constant high gain. Actually, the peaking values for \( \dot{x}_2, \dot{x}_3 \) are the orders of \( r_0, r_0^2 \), respectively. The larger \( r_0 \) is, the larger the peaking values are.

Next, we apply the time varying gain and let the gain be relatively small in the initial stage. Then the initial value of error \( \eta \) is
\[
(r(0)^2(x_1(0) - \hat{x}_1(0)), r(0)(x_2(0) - \hat{x}_2(0)), x_3(0) - \hat{x}_3(0))^\top,
\tag{2.33}
\]
which is also small. Actually if \( r(0) = 1 \), then the initial value of error \( \eta \) is
\[
\begin{pmatrix}
(x_1(0) - \hat{x}_1(0)) & (x_2(0) - \hat{x}_2(0)) & x_3(0) - \hat{x}_3(0)
\end{pmatrix}^\top.
\tag{2.34}
\]
Finding the derivative of $V$ along (2.31) we can obtain that if $\|\eta(t)\| \geq 1$ then
\[
\left. \frac{dV}{dt} \right|_{\text{along } (2.31)} \leq -r(t)\|\eta(t)\|^2 + 2\lambda_{\max}(P) \sup_{t \in [0, \infty)} |\dot{\eta}(t)||\eta(t)| \\
\leq - (r(t) - 2M\lambda_{\max}(P)) \|\eta(t)\|^2.
\] (2.35)

It is seen that when $r$ increases to $2M\lambda_{\max}(P)$, then $V(\eta(t))$ stops increasing. This together with
\[
\|\eta(t)\| \leq \frac{1}{\lambda_{\max}(P)} V(\eta(t))
\] (2.36)
shows that $\|\eta\|$ does not increase any more although $r$ increases continuously to a large number $r_0$ or $\infty$. If $r(0) = 1$ and $M < 1/(2\lambda_{\max}(P))$, then $\|\eta(t)\| \leq 1$ uniformly on $[0, \infty)$. It follows from the definition of $\eta$ that the peaking values of $\hat{x}_2$ and $\hat{x}_3$ in this case are quite small. In fact,
\[
\sup_{t \in [0, \infty)} |\hat{x}_1(t)| \leq 1 + \sup_{t \in [0, \infty)} |x_i(t)|,
\] (2.37)

no matter how large the $r$ reaches at last. While by using the constant gain $r_0$ in this case, the peaking value of $\hat{x}_2$ is about $B_1r_0$, and the peaking value of $\hat{x}_3$ is about $B_2r_0^2$, where $B_1$ and $B_2$ depend on the matrix $E$ and initial values $x_i(0), i = 1, 2$ and $\hat{x}_j(0), j = 1, 2, 3$.

### 3 Proof of main results

**Proof of Theorem 2.1.** Let
\[
\eta_i(t) = r^{n+1-i}(t) (x_i(t) - \hat{x}_i(t)), i = 1, \ldots, n+1, \\
\mu_j(t) = x_j(t) - z_j(t), j = 1, \ldots, n, \\
\eta(t) = (\eta_1(t), \ldots, \eta_{n+1}(t))^\top, \mu(t) = (\mu_1(t), \ldots, \mu_n(t))^\top.
\] (3.38)

In $[0, \frac{1}{r} \ln r_0)$, the errors satisfy
\[
\begin{cases}
\dot{\eta}_1(t) = r(t)(\eta_2(t) - g_1(\eta_1(t))) + \frac{\eta_1(t)}{r(t)} \eta_1(t), \\
\vdots \\
\dot{\eta}_n(t) = r(t)(\eta_{n+1}(t) - g_n(\eta_1(t))) + \frac{\eta_n(t)}{r(t)} \eta_n(t), \\
\dot{\eta}_{n+1}(t) = -r(t)g_{n+1}(\eta_1(t)) + \hat{x}_{n+1}(t), \\
\dot{\mu}(t) = A_n\mu(t) + B_n[u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) + \eta_{n+1}(t)].
\end{cases}
\] (3.39)

while in $[\frac{1}{r} \ln r_0, \infty)$,
\[
\begin{cases}
\dot{\eta}_1(t) = r_0(\eta_2(t) - g_1(\eta_1(t))), \\
\vdots \\
\dot{\eta}_n(t) = r_0(\eta_{n+1}(t) - g_n(\eta_1(t))), \\
\dot{\eta}_{n+1}(t) = -r_0g_{n+1}(\eta_1(t)) + \hat{x}_{n+1}(t), \\
\dot{\mu}(t) = A_n\mu(t) + B_n[u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) + \eta_{n+1}(t)].
\end{cases}
\] (3.40)
where $A_0$ and $B_0$ are the same as in (1.2). Since $u_0, g_i$ are continuous and $r$ is also continuous in \([0, \frac{\ln r_0}{a}]\), Equation (3.40) admits a continuous solution in \([0, \frac{\ln r_0}{a}]\). Similarly, by the continuity of $u_0$ and $g_i$’s, Equation (3.40) admits a continuous solution as well in \([\frac{\ln r_0}{a}, \infty)\).

We first need to estimate the derivative of the total disturbance. By (1.1), (2.6), (1.4), and (3.38),

$$
\dot{x}_n(t) = f(t, x(t), \zeta(t), w(t)) + b(t)u(t)
= x_{n+1}(t) + z_{n+1}(t) - \dot{x}_{n+1}(t) + u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t))
= u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t)) + \eta_{n+1}(t) + z_{n+1}(t).
$$

By (1.4), (1.1), (1.3), and (3.41),

$$
\dot{x}_{n+1}(t) = \frac{d}{dt}[f(t, x(t), \zeta(t), w(t)) + (b(t) - b_0)u(t)]
= \frac{\partial f}{\partial t}(t, x(t), \zeta(t), w(t)) + \sum_{i=1}^{n-1} x_{i+1}(t) \frac{\partial f}{\partial x_i}(t, x(t), \zeta(t), w(t))
+ |u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t)) + \eta_{n+1}(t)| \frac{\partial f}{\partial x_n}(t, x(t), \zeta(t), w(t))
+ f_0(t, x(t), \zeta(t), w(t)) \cdot \frac{\partial f}{\partial \zeta}(t, x(t), \zeta(t), w(t)) + \dot{w}(t) \frac{\partial f}{\partial w}(t, x(t), \zeta(t), w(t))
+ \left(\frac{b(t) - b_0}{b_0}\right) \left[ u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t)) + \eta_{n+1}(t) - \dot{x}_{n+1}(t) \right]
+ \left(\frac{b(t) - b_0}{b_0}\right) \left[ \sum_{i=1}^{n} \left( \dot{x}_{i+1}(t) + \frac{1}{r^{i+1}(t)}g_i(\eta_i(t)) - z_i(t) \right) \right],
$$

where $\frac{\partial u_0}{\partial \eta_i}$ is the partial derivative of $u_0$ with respect to its $i$-th component.

From Assumption A1 and (3.38),

$$
\left| \frac{\partial f}{\partial t}(t, x(t), \zeta(t), w(t)) \right| \leq K_1 + K_2\|x(t) - z(t)\| + \|z(t)\| + \varpi(w(t))
\leq (K_1 + K_2M_1 + \varpi(w(t))) + K_2\|\mu(t)\|,
$$

where $z(t) = (z_1(t), \ldots, z_n(t))^\top$. Similarly,

$$
\left| \sum_{i=1}^{n-1} x_{i+1}(t) \frac{\partial f}{\partial x_i}(t, x(t), \zeta(t), w(t)) \right| \leq (K_0 + \varpi(w(t))) \sum_{i=1}^{n-1} |x_{i+1}(t)|
\leq n(K_0 + \varpi(w(t)))\|\mu(t)\| + nM_1(K_0 + \varpi(w(t))),
$$

$$
\left| f_0(t, x(t), \zeta(t), w(t)) \cdot \frac{\partial f}{\partial w}(t, x(t), \zeta(t), w(t)) \right| \leq (K_1 + K_2\|x(t)\| + \varpi(w(t)))(K_0 + \varpi(w(t)))
\leq (K_0 + \varpi(w(t)))(K_1 + \varpi(w(t)) + K_2M_1) + (K_0 + \varpi(w(t)))K_2\|\mu(t)\|,
$$

(3.43)

(3.44)

(3.45)
and
\[
\left| \dot{w}(t) \frac{\partial L}{\partial \zeta}(t, x(t), \zeta(t), w(t)) \right| \leq M_1(K_1 + \varpi(w(t)) + K_2\left\| x(t) \right\|) \\
\leq M_1(K_1 + K_2M_1 + \varpi(w(t))) + M_1K_2\|\mu(t)\|. \tag{3.46}
\]

Since by Assumption A3, \( r \geq 1 \). It follows from (3.38) that
\[
\left\| (\hat{x}_1(t) - x_1(t), \ldots, \hat{x}_n(t) - x_n(t)) \right\| = \left\| \left( \frac{m(t)}{r^n(t)}, \ldots, \frac{m(t)}{r^n(t)} \right) \right\| \leq \|\eta(t)\|. \tag{3.47}
\]

By Assumption A1 and (3.47),
\[
\sum_{i=1}^{n+1} \left| \hat{x}_{i+1}(t) + \frac{1}{r^{n-i}(t)} \right| g_i(\eta_i(t)) - z_{i+1} \right| \\
\leq \sum_{i=2}^{n} |\hat{x}_i(t) - x_i(t)| + \sum_{i=2}^{n} |x_i(t) - z_i(t)| + |x_{n+1}(t)| + |\eta_{n+1}(t)| + \sum_{i=1}^{n} \frac{1}{r^{n-i}(t)} \left| g_i(\eta_i(t)) \right| \tag{3.48}
\]

\[
\leq n(\|\mu(t)\| + \|\eta(t)\|) + \sum_{i=1}^{n} |g_i(\eta_i(t))| + M_1 + |x_{n+1}(t)|.
\]

Furthermore, by (1.4) and (2.6),
\[
x_{n+1}(t) = f(t, x(t), \zeta(t), w(t)) + (b(t) - b_0)u(t) \\
= f(t, x(t), \zeta(t), w(t)) + \frac{b(t) - b_0}{b_0} u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) \tag{3.49}
+ \frac{b(t) - b_0}{b_0} \left( (x_{n+1}(t) - \hat{x}_{n+1}(t)) + z_{n+1}(t) \right) - \frac{b(t) - b_0}{b_0} x_{n+1}(t).
\]

It follows that
\[
x_{n+1}(t) = \frac{b_0}{b(t)} f(t, x(t), \zeta(t), w(t)) + \frac{b(t) - b_0}{b(t)} u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) \\
+ \left( x_{n+1}(t) - \hat{x}_{n+1}(t) \right) + z_{n+1}(t). \tag{3.50}
\]

Since \( |(b(t) - b_0)/b_0| \leq 1/2 \) by assumption, \( b_0/2 \leq b(t) \leq 3b_0/2, \| (b(t) - b_0)/(b(t)) \| \leq 1 \). This together with (3.50) yields
\[
|x_{n+1}(t)| \leq 2(K_1 + K_2\| x(t) \| + \varpi(w(t))) + |u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) + \|\eta(t)\| + M_1 \\
\leq M_1 + 2(K_1 + K_2M_1 + \varpi(w(t))) + 2K_2\|\mu(t)\| + \|\eta(t)\| \\
+ |u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t))|, \tag{3.51}
\]

where
\[
u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) = u_0(\mu_1(t), \ldots, \mu_n(t)) \\
+ \left( u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) - u_0(x_1(t) - z_1(t), \ldots, x_n(t) - z_n(t)) \right). \tag{3.52}
\]

By assumption again, all partial derivatives of \( u_0 \) are bounded by \( L \), we have
\[
\left| u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) - u_0(x_1(t) - z_1(t), \ldots, x_n(t) - z_n(t)) \right| \\
\leq |u_0(\hat{x}_1(t) - z_1(t), \hat{x}_2(t) - z_2(t), \ldots, \hat{x}_n(t) - z_n(t)) - u_0(x_1(t) - z_1(t), \ldots, x_n(t) - z_n(t))| \leq L M_{b_0} \|\mu(t)\|,
\]

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The remainder of the proof is split into the following three steps.

By (3.42)-(3.54) and Assumption A1, there exists $B > 0$ such that for all $t \in [0, \infty)$,

$$\left| \dot{x}_{n+1}(t) \right| \leq B \left( 1 + \|\eta(t)\| + \|\mu(t)\| + \sum_{i=1}^{n} |g_i(\eta(t))| + \mu(t) \right) \frac{b(t) - b_0}{b_0} r_0 |g_{n+1}(\eta(t))|. \quad (3.55)$$

Let $\mathcal{V}, W, V$ and $W$ be the Lyapunov functions satisfying Assumptions A2 and A3. Define the Lyapunov functions $\mathcal{W}, \mathfrak{W} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^+$:

$$\mathcal{W}(x, y) = \mathcal{V}(x) + V(y), \quad \mathfrak{W}(x, y) = W(x) + W(y). \quad (3.56)$$

Finding the derivatives of $\mathfrak{W}$ along the solution of (3.40) gives

$$\frac{d\mathfrak{W}(\eta(t), \mu(t))}{dt} \bigg|_{(3.40)} = r_0 \left( \sum_{i=1}^{n} \left( \eta_{i+1}(t) - g_i(\eta(t)) \right) \frac{\partial \mathcal{V}}{\partial \eta_i}(\eta(t)) - g_{n+1}(\eta(t)) \frac{\partial \mathcal{V}}{\partial \eta_{n+1}}(\eta(t)) \right)$$

$$+ \sum_{i=1}^{n} \mu_{i+1}(t) \frac{\partial \mathcal{V}}{\partial \mu_i}(\mu(t)) + \left[ u_0(\dot{x}_1(t) - z_1(t), \ldots, \dot{x}_n(t) - z_n(t)) + \eta_{n+1}(t) \right] \frac{\partial \mathcal{V}}{\partial \eta_{n+1}}(\eta(t))$$

$$+ \dot{x}_{n+1}(t) \frac{\partial \mathcal{V}}{\partial \eta_{n+1}}(\eta(t)). \quad (3.57)$$

Let

$$r_1 = \max \left\{ \frac{4B(n+1)N_1}{\Delta}, \frac{2(B + (n + 1)L)N_1N_2}{\Delta} \right\}, \quad (3.58)$$

where

$$\Delta = 1 - N_0 \sup_{t \in [0, \infty)} \|(b(t) - b_0)/b_0\|. \quad (3.59)$$

By Assumptions A2 and A3, (3.57), (3.55), (3.59), (3.53), and (3.58), for any $r_0 > r_1$ and $t > (\ln r_0)/a$,

$$\frac{d\mathfrak{W}(\eta(t), \mu(t))}{dt} \bigg|_{(3.40)} \leq -\Delta r_0 W(\eta(t)) + B \sqrt{N_1} \sqrt{W(\eta(t))} - W(\mu(t))$$

$$+ B(n+1)N_1 W(\eta(t)) + (B + (n + 1)L) \sqrt{N_1N_2} \sqrt{W(\eta(t))} \sqrt{W(\mu(t))}$$

$$\leq -\Delta r_0 W(\eta(t))/2 + B \sqrt{N_1} \sqrt{W(\eta(t))} - W(\mu(t))/2. \quad (3.60)$$

The remainder of the proof is split into the following three steps.
Step 1. The boundedness of $\|\langle \eta(t), \mu(t) \rangle\|$. Let

$$R = \max \left\{ \kappa_3^{-1} \left( \frac{(4BN_1^{1/2}}{\Delta r_0} \right)^2, \kappa_3^{-1} (4B^2(N_1N_3)) \right\}. \quad (3.61)$$

If $\|\langle \eta(t), \mu(t) \rangle\| > 2R$, then there are two cases. When $\|\eta(t)\| > R$ and $r > r_1$,

$$\left. \frac{d\mathcal{W}(\eta(t), \mu(t))}{dt} \right|_{(3.40)} \leq -\frac{\Delta}{2} r_0 W(\eta(t)) + B\sqrt{N_1} \sqrt{W(\eta(t))}$$

$$\leq \sqrt{W(\eta(t))} \left( -\frac{\Delta}{2} r_0 \kappa_3(\|\eta(t)\|) + B\sqrt{N_1} \right) \leq -B\sqrt{N_1} \sqrt{\kappa_3(R)} < 0. \quad (3.62)$$

When $\|\eta(t)\| < R$, $\|\mu(t)\| > R$, it follows from (2.9) that

$$\left. \frac{d\mathcal{W}(\eta(t), \mu(t))}{dt} \right|_{(3.39)} \leq B\sqrt{N_1} \sqrt{W(\eta(t))} - \frac{1}{2} W(\mu(t))$$

$$\leq B\sqrt{N_1N_3} \sqrt{\kappa_4(\|\eta(t)\|)} - \frac{1}{2} \kappa_3(\|\mu(t)\|) \leq B\sqrt{N_1N_3} \sqrt{\kappa_3(R)} - \frac{1}{2} \kappa_3(R)$$

$$\leq -B\sqrt{N_1N_3} \sqrt{\kappa_3(R)} < 0, \forall \ r_0 > r_1. \quad (3.63)$$

From aforementioned facts, there exists a $t_1 > (\ln r_0)/a$ such that $\|\langle \eta(t), \mu(t) \rangle\| \leq 2R$ for all $t > t_1$.

Step 2. Convergence of $\eta(t)$. Since $\langle \eta(t), \mu(t) \rangle$ is uniformly bounded for $t > t_1$, it follows from the continuity of $g_i$’s, $\nabla V$, and (3.55) that there exists a $B_1 > 0$ such that for all $t > t_1$,

$$|\dot{x}_{n+1}(t)| \left| \frac{\partial V}{\partial \eta_{n+1}}(\eta(t)) \right| \leq B_1 + N_0 \sup_{t \in [0, \infty)} |(b(t) - b_0)/b_0| r_0 W(\eta(t)). \quad (3.64)$$

By (3.64) and Assumption A2, we find the derivative of $V(\eta)$ along the solution of (3.40) to be

$$\frac{V(\eta(t))}{dt} \bigg|_{(3.39)} = r_0 \left( \sum_{i=1}^{n} (\eta_{i+1}(t) - g_i(\eta_i(t))) \frac{\partial V}{\partial \eta_i}(\eta(t)) - g_{n+1}(\eta_1(t)) \frac{\partial V}{\partial \eta_{n+1}}(\eta(t)) \right)$$

$$+ \dot{x}_{n+1}(t) \frac{\partial V}{\partial \eta_{n+1}}(\eta(t)) \leq -\Delta r_0 W(\eta(t)) + B_1. \quad (3.65)$$

For any given $\sigma > 0$, let

$$r^* = \max \left\{ r_1, \frac{2B_1}{\Delta \kappa_3(\sigma)} \right\}. \quad (3.66)$$

For any $r_0 > r^*$, if $\|\eta(t)\| > \sigma$, then $W(\eta(t)) \geq \kappa_3(\eta(t)) \geq \kappa_3(\sigma)$. It follows that

$$\frac{V(\eta(t))}{dt} \bigg|_{(3.39)} \leq -B_1 < 0. \quad (3.67)$$

So there exists a $t_\sigma > t_1$ such that $\|\eta(t)\| < \sigma$ for all $t > t_\sigma$.

Step 3. Convergence of $\mu(t)$. From Step 1, $\mu(t)$ is uniformly bounded for $t > t_1$. This together with the continuity of $\nabla V$ shows that there exists a constant $C_1 > 0$ such that $(\ln N + 1) \left| \frac{\partial V}{\partial \mu_i}(\mu(t)) \right| < C_1$ for all $t > t_1$. Hence the derivative of $V$ along the solution of (3.40) satisfies

$$\left. \frac{dV(\mu(t))}{dt} \right|_{(3.40)} = \sum_{i=1}^{n-1} \mu_{i+1}(t) \frac{\partial V}{\partial \mu_i}(\mu(t))$$
$$+[u_0(\hat{x}_1(t) - z_1(t), \ldots, \hat{x}_n(t) - z_n(t)) + \eta_{n+1}(t)] \frac{\partial V}{\partial \mu_n+1}(\mu(t)) \leq -W(\mu(t)) + C_1\|\eta(t)\|. \quad (3.68)$$

For any $\sigma > 0$, by Step 2, one can find a $t_{\sigma 1} > t_1$ such that $\|\eta(t)\| < \frac{1}{2\kappa_3} \forall t > t_{\sigma 1}$. It then follows that for any $t > t_{\sigma 1}$, if $\|\mu(t)\| > \sigma$, then

$$\frac{dV(\mu(t))}{dt}\bigg|_{(3.40)} \leq -W(\mu(t)) + C_1\|\eta(t)\| \leq -\kappa_3(\sigma) + C_1\|\eta(t)\| \leq -\frac{1}{2}\kappa_3(\sigma) < 0. \quad (3.69)$$

Therefore one can find a $t_{\sigma 2} > t_{\sigma 1}$ such that $\|\mu(t)\| < \sigma$ for $t > t_{\sigma 2}$.

**Proof of Theorem 2.2.**

Let $\eta_i$ be defined in (3.38) with $x_{n+1} = \tilde{f}(t, \zeta(t), w(t))$. A straightforward computation shows that for every $\left[\frac{\ln r_0}{a}, \infty \right)$

$$\begin{cases}
\dot{\eta}_1(t) = r_0(\eta_2(t) - g_1(\eta_1(t))), \\
\vdots \\
\dot{\eta}_n(t) = r_0(\eta_{n+1}(t) - g_n(\eta_1(t))) + r_0[\tilde{f}(x_1, \ldots, x_n) - \tilde{f}(\hat{x}_1, \ldots, \hat{x}_n)], \\
\dot{\eta}_{n+1}(t) = -r_0 g_{n+1}(\eta_1(t)) + \hat{x}_{n+1}(t).
\end{cases} \quad (3.70)$$

From the second condition of Assumption A1*, there exists a positive constant $B$ such that

$$|\hat{x}_{n+1}(t)| = \left| \frac{d}{dt}\tilde{f}(t, \zeta(t), w(t)) \right| < B, \ t > 0. \quad (3.71)$$

By the first condition of Assumption A1* and (3.38),

$$|\tilde{f}(x_1, \ldots, x_n) - \tilde{f}(\hat{x}_1, \ldots, \hat{x}_n)| \leq L \left\| \left( \frac{\eta_1}{r_0^0}, \ldots, \frac{\eta_n(t)}{r_0} \right) \right\| \leq \frac{L}{r_0^0} \|\eta_1, \ldots, \eta_n\|^\beta. \quad (3.72)$$

Let $V$, $W$ be the Lyapunov functions satisfying Assumption A2*. Then there exists class $K_\infty$ functions $\kappa_i, i = 1, 2, 3, 4$ satisfying (2.9). By Assumption A2*, (3.71), and (3.72), we can find the derivative of $V$ along the error equation (3.70) to obtain that for any $t > t_1$, if $\|\eta(t)\| > \overline{R}$, then

$$\frac{dV(\eta(t))}{dt}\bigg|_{(3.70)} = r_0 \left( \sum_{i=1}^n (\eta_{i+1}(t) - g_i(\eta_1(t))) \frac{\partial V}{\partial \eta_i}(\eta(t)) - g_{n+1}(\eta_1(t)) \frac{\partial V}{\partial \eta_{n+1}}(\eta(t)) \right)$$

$$+r_0[\tilde{f}(x_1, \ldots, x_n) - \tilde{f}(\hat{x}_1, \ldots, \hat{x}_n)] \frac{\partial V}{\partial \eta_n}(\eta(t)) + \hat{x}_{n+1}(t) \frac{\partial V}{\partial \eta_{n+1}}(\eta(t)) \leq -r_0V(\eta(t)) + LN_0^{1-\beta}W(\eta(t)) + BNW(\eta(t)). \quad (3.73)$$

By (2.9), (3.73), for any

$$r_0 > \max \left\{ 2BN, (2LN)^{1/\beta} \right\}. \quad (3.74)$$

and $t > \frac{\ln r_0}{a}$, if $\|\mu(t)\| > \overline{R}$, then

$$\frac{dV(\eta(t))}{dt}\bigg|_{(3.70)} \leq -BN\kappa_3(\overline{R}) < 0, \quad (3.75)$$
which shows that the solution $\eta$ of (3.70) is uniformly bounded for all $t \geq t_1$ for some $t_1 > \frac{\ln r_0}{a}$. Therefore one can find a constant $C > 0$ such that
\[
\frac{dV(\eta(t))}{dt} \bigg|_{(3.70)} \leq -\frac{1}{2} r(t) W(\eta(t)) + C, \quad \forall \ t > t_1.
\] (3.76)

The remainder proof is similar to the proof of Theorem 2.1, the details are omitted.

4 Concluding remarks

In this paper, we propose an modified nonlinear extended state observer (ESO) for a quite general class of nonlinear systems with large uncertainties from dynamics, control, and external disturbance. A time-varying gain function is introduced in ESO. The convergence of active disturbance rejection control (ADRC) for this kind of system is investigated. First, a practical asymptotic stability of ESO is developed. Second, the closed-loop system under the ESO-based output feedback control is shown to be practically asymptotically stable to zero equilibrium by rejecting the total disturbance. This result can be regarded as a kind of separation principle for this special control strategy. Owing to its estimation/cancelation nature, the control energy in ADRC can be reduced significantly as reviewed elsewhere [32]. Numerical simulations demonstrate that the modified ESO can diminish effectively the peaking value happened only in the initial stage, a notorious problem caused by constant high gain.

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References


