

On Stability Analysis of Active Disturbance Rejection Control for Nonlinear Time-Varying Plants with Unknown Dynamics

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Abstract—This paper concerns with stability characteristics of active disturbance rejection control for nonlinear time-varying plants that are largely unknown. In particular, asymptotic stability is established where the plant dynamics is completely known. In the face of large dynamic uncertainties, estimation and tracking errors are shown to be bounded, with their bounds monotonously decreasing with their respective bandwidths.

Index Terms—Linear active disturbance rejection control, linear extended state observer, uncertain systems, stability.

I. INTRODUCTION

Most physical plants in real world are not just nonlinear and time-varying but also highly uncertain. Control system design for such systems has been the focus of much of the recent developments under the umbrella of robust, adaptive, and nonlinear control. Most of the existing results, however, are obtained presupposing that a fairly detailed and accurate mathematical model of the plant is available. The small gain theorem based robustness analysis does allow a small amount of uncertainties in plant dynamics, but not anywhere near the magnitude often encountered in practice. As the well-known control theorist Roger Brockett puts it: “If there is no uncertainty in the system, the control, or the environment, feedback control is largely unnecessary” [1]. The assumption that a physical plant, without feedback, behaves rather closely as its mathematical model describes, as the point of departure in control system design, does not reflect either the intent of feedback control, or the physical reality.

The active disturbance rejection control (ADRC) was proposed as an alternative paradigm to address this fundamental issue [2]. The main difference in the design concept pertains to the question of how much model information is needed. Recognizing the vulnerability of the reliance on accurate mathematical model, there has been a gradual recognition over the years that active disturbance estimation is a viable alternative to an accurate plant model. That is, if the disturbance, representing the discrepancy between the plant and its model, is estimated in real time, then the plant-model mismatch can be effectively compensated for, making the model based design tolerant of a large amount of uncertainties. The focal point is how external disturbance and unknown dynamics can be estimated. Several classes of approach are outlined below, including the unknown input

observer (UIO) [3]-[10], the disturbance observer (DOB) [11]-[18], the perturbation observer (POB) [19]-[22], and the extended state observer (ESO) [2], [23]-[24].

UIO is the earliest disturbance estimator, going back to 1969, where the external disturbance is formulated as an augmented state and estimated using a state observer. DOB is another main class of disturbance estimators, based on the inverse of the nominal transfer function of the plant, which is avoided in some of the variations of DOB [25]-[26]. POB is another class of disturbance estimators, similar to DOB in concept but formulated in state space in discrete time domain.

It was shown that UIO and DOB are equivalent in terms of disturbance estimation [16] but UIO also provides estimation of the states. Similar to UIO, ESO is also a state space approach. What sets ESO apart from UIO and DOB is that it is conceived to estimate not only the external disturbance but also plant dynamics. Among the disturbance estimators, ESO requires the least amount of plant information.

All estimators above, including UIO, DOB, POB and ESO, prove to be effective practical solutions. But how fast and in what range the disturbance and unknown dynamics can be estimated are not obvious. In particular, both UIO and DOB are originally formulated to estimate the external disturbances but later adopted to estimate the unknown plant dynamics as well, with very little analytical support on how this can be achieved. Even when limited stability analysis was performed, only boundedness of estimation or compensation error was obtained, while the actual bound of the error is largely unknown. Some robust stability analysis, based on small gain theorem, was performed for UIO and DOB [16]-[18] but the results tend to be quite conservative by nature and limited to linear and time-invariant plants. For nonlinear plants, only limited results on stability properties are obtained for robot manipulators [12], [15].

In a rare exception, the approach proposed by [27] is designed specifically to estimate both unknown dynamics and disturbance with asymptotic stability of the closed-loop system firmly established. But the practicality method is quickly called into question as the observer, hence the stability proof, requires the use of higher order derivatives of the output, rendering the system susceptible to noise corruption. In short, for those effective practical solutions, there seems to be a lack of rigorous analysis of the estimation error, especially its bound. But for those methods firmly rooted in mathematical rigor, the utility is often questionable. This paper specifically addresses this issue, pertaining to ESO and the associated ADRC.

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ADRC, proposed by Han in 1995, is designed to deal with those plants with large amount of uncertainties both in dynamics and external disturbances [23]. It was further simplified to linear ADRC (LADRC), using linear ESO (LESO) in [24], which makes it extremely simple and practical [28]. The purpose of this paper is to show analytically how LADRC achieves excellent performance, even when the plant is unknown, nonlinear and time-varying. The convergence and the bounds of the estimation and tracking errors are presented. In particular, at one extreme, the asymptotic stability of LADRC is proved where the accurate mathematical model of the plant is given. At the other extreme, where the plant dynamics is largely unknown, the upper bounds of errors are derived.

The paper is organized as follows. The analyses for the LESO and for LADRC are presented in Section II and Section III respectively, with and without a detailed mathematical model. The paper ends with a few concluding remarks in Section IV.

II. ANALYSIS OF LESO ERROR DYNAMICS

Consider a generally nonlinear time-varying dynamic system with single-input, u , and single-output y ,

$$y^{(n)}(t) = f(y^{(n-1)}(t), \dots, y(t), w(t)) + bu(t). \quad (1)$$

where w is the external disturbance and b is a given constant. Here $f(y^{(n-1)}(t), y^{(n-2)}(t), \dots, y(t), w(t))$, or simply denoted as f , represents the nonlinear time-varying dynamics of the plant that is unknown. That is, for this plant, only the order and the parameter b are given. The ADRC is a unique method designed to tackle this problem. It is centered around estimation of, and compensation for, f . To this end, assuming f is differentiable and let $h = \dot{f}$, (1) can be written in an augmented state space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= x_{n+1} + bu \\ \dot{x}_{n+1} &= h(x, w) \\ y &= x_1 \end{aligned} \quad (2)$$

where $x = [x_1, x_2, \dots, x_{n+1}]^T \in \mathbb{R}^{n+1}$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state, input and output of the system, respectively. Any state observer of (2), will estimate the derivatives of y and f since the latter is now a state in the extended state model. Such observers are known as ESO. The convergence of the estimation error for a particular ESO, LESO, is shown below.

A. Convergence of the LESO with the Given Model of the Plant

With u and y as inputs and the function h given, the LESO of (2) is given as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + l_{n-1}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_n &= \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu \\ \dot{\hat{x}}_{n+1} &= l_{n+1}(x_1 - \hat{x}_1) + h(\hat{x}, w) \end{aligned} \quad (3)$$

where $\hat{x} = [\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{n+1}]^T \in \mathbb{R}^{n+1}$, and $l_i, i = 1, 2, \dots, n+1$, are the observer gain parameters to be chosen. In particular, let us consider a special case where the gains are chosen as

$$[l_1, l_2, \dots, l_{n+1}] = [\omega_o \alpha_1, \omega_o^2 \alpha_2, \dots, \omega_o^{n+1} \alpha_{n+1}] \quad (4)$$

with $\omega_o > 0$. Here $\alpha_i, i = 1, 2, \dots, n+1$, are selected such that the characteristic polynomial $s^{n+1} + \alpha_1 s^n + \dots + \alpha_n s + \alpha_{n+1}$ is Hurwitz. For simplicity, let $s^{n+1} + \alpha_1 s^n + \dots + \alpha_n s + \alpha_{n+1} = (s+1)^{n+1}$ where $\alpha_i = \frac{(n+1)!}{i!(n+1-i)!}$, $i = 1, 2, \dots, n+1$. Then the characteristic polynomial of (3) is

$$\lambda_o(s) = (s + \omega_o)^n. \quad (5)$$

and ω_o , the observer bandwidth, becomes the only tuning parameter of the observer.

Let $\tilde{x}_i = x_i - \hat{x}_i, i = 1, 2, \dots, n+1$. From (2) and (3), the observer estimation error can be shown as

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 - \omega_o \alpha_1 \tilde{x}_1 \\ &\vdots \\ \dot{\tilde{x}}_{n-1} &= \tilde{x}_n - \omega_o^{n-1} \alpha_{n-1} \tilde{x}_1 \\ \dot{\tilde{x}}_n &= \tilde{x}_{n+1} - \omega_o^n \alpha_n \tilde{x}_1 \\ \dot{\tilde{x}}_{n+1} &= h(x, w) - h(\hat{x}, w) - \omega_o^{n+1} \alpha_{n+1} \tilde{x}_1. \end{aligned} \quad (6)$$

Now let $\varepsilon_i = \frac{\tilde{x}_i}{\omega_o^{i-1}}, i = 1, 2, \dots, n+1$, then (6) can be rewritten as

$$\dot{\varepsilon} = \omega_o A \varepsilon + B \frac{h(x, w) - h(\hat{x}, w)}{\omega_o^n} \quad (7)$$

where $A = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_n & 0 & \cdots & 0 & 1 \\ -\alpha_{n+1} & 0 & \cdots & 0 & 0 \end{bmatrix}$, $B = [0 \ 0 \ \cdots \ 0 \ 1]^T$. Here A is Hurwitz for the $\alpha_i, i = 1, 2, \dots, n+1$, chosen above.

Theorem 1: Assuming $h(x, w)$ is globally Lipschitz with respect to x , there exists a constant $\omega_o > 0$, such that $\lim_{t \rightarrow \infty} \tilde{x}_i(t) = 0, i = 1, 2, \dots, n+1$.

Proof. Since A is Hurwitz, there exists a unique positive definite matrix P such that $A^T P + PA = -I$. Choose the Lyapunov function as $V(\varepsilon) = \varepsilon^T P \varepsilon$. Hence

$$\dot{V}(\varepsilon) = -\omega_o \|\varepsilon\|^2 + 2\varepsilon^T P B \frac{h(x, w) - h(\hat{x}, w)}{\omega_o^n}. \quad (8)$$

Since the function $h(x, w)$ is globally Lipschitz with respect to x , that is, there exists a constant c' such that $|h(x, w) - h(\hat{x}, w)| \leq c' \|x - \hat{x}\|$ for all x, \hat{x} , and w , it follows that

$$2\varepsilon^T PB \frac{|h(x, w) - h(\hat{x}, w)|}{\omega_o^n} \leq 2\varepsilon^T PBc' \frac{\|x - \hat{x}\|}{\omega_o^n}. \quad (9)$$

When $\omega_o \geq 1$, one has $\frac{\|x - \hat{x}\|}{\omega_o^n} = \frac{\|\tilde{x}\|}{\omega_o^n} = \frac{\|\sqrt{\varepsilon_1^2 + \varepsilon_2^2 \omega_o^2 + \varepsilon_3^2 \omega_o^4 + \dots + \varepsilon_{n+1}^2 \omega_o^{2n}}\|}{\omega_o^n} \leq \|\varepsilon\|$. Therefore, we obtain

$$2\varepsilon^T PB \frac{|h(x, w) - h(\hat{x}, w)|}{\omega_o^n} \leq c \|\varepsilon\|^2 \quad (10)$$

where $c = 1 + \|PBc'\|^2$. From (8) and (10), one has

$$\dot{V}(\varepsilon) \leq -(\omega_o - c) \|\varepsilon\|^2. \quad (11)$$

That is, $\dot{V}(\varepsilon) < 0$ if $\omega_o > c$. Therefore, $\lim_{t \rightarrow \infty} \tilde{x}_i(t) = 0, i = 1, 2, \dots, n+1$, for $\omega_o > c$. Q.E.D.

B. Convergence of the LESO with Plant Dynamics Largely Unknown

In many real world scenarios, the plant dynamics represented by f is mostly unknown. In this case, the LESO in (3) now takes the form of

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + l_{n-1}(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_n &= \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu \\ \dot{\hat{x}}_{n+1} &= l_{n+1}(x_1 - \hat{x}_1). \end{aligned} \quad (12)$$

Consequently, the observer estimation error in (6) becomes

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 - \omega_o \alpha_1 \tilde{x}_1 \\ &\vdots \\ \dot{\tilde{x}}_{n-1} &= \tilde{x}_n - \omega_o^{n-1} \alpha_{n-1} \tilde{x}_1 \\ \dot{\tilde{x}}_n &= \tilde{x}_{n+1} - \omega_o^n \alpha_n \tilde{x}_1 \\ \dot{\tilde{x}}_{n+1} &= h(x, w) - \omega_o^{n+1} \alpha_{n+1} \tilde{x}_1 \end{aligned} \quad (13)$$

and Equation (7) is now

$$\dot{\varepsilon} = \omega_o A \varepsilon + B \frac{h(x, w)}{\omega_o^n}. \quad (14)$$

Theorem 2: Assuming $h(x, w)$ is bounded, there exist a constant $\sigma_i > 0$ and a finite $T_1 > 0$ such that $|\tilde{x}_i(t)| \leq \sigma_i, i = 1, 2, \dots, n+1, \forall t \geq T_1 > 0$ and $\omega_o > 0$. Furthermore, $\sigma_i = O\left(\frac{1}{\omega_o^k}\right)$, for some positive integer k .

Proof. Solving (14), it follows that

$$\varepsilon(t) = e^{\omega_o A t} \varepsilon(0) + \int_0^t e^{\omega_o A(t-\tau)} B \frac{h(x(\tau), w)}{\omega_o^n} d\tau. \quad (15)$$

Let

$$p(t) = \int_0^t e^{\omega_o A(t-\tau)} B \frac{h(x(\tau), w)}{\omega_o^n} d\tau, \quad (16)$$

since $h(x(\tau), w)$ is bounded, that is, $|h(x(\tau), w)| \leq \delta$, where δ is a positive constant, for $i = 1, 2, \dots, n+1$, we have

$$|p_i(t)| \leq \frac{\delta}{\omega_o^{n+1}} [|(A^{-1}B)_i| + |(A^{-1}e^{\omega_o A t}B)_i|]. \quad (17)$$

For A and B defined in (7), $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & -\frac{1}{\alpha_{n+1}} \\ 1 & 0 & 0 & \dots & -\frac{\alpha_1}{\alpha_{n+1}} \\ 0 & 1 & 0 & \dots & -\frac{\alpha_2}{\alpha_{n+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{\alpha_n}{\alpha_{n+1}} \end{bmatrix}$, and

$$|(A^{-1}B)_i| \leq \nu \quad (18)$$

where $\nu = \max_{i=2, \dots, n+1} \left\{ \frac{1}{\alpha_{n+1}}, \frac{\alpha_{i-1}}{\alpha_{n+1}} \right\}$. Since A is Hurwitz, there exists a finite time $T_1 > 0$ such that

$$|[e^{\omega_o A t}]_{ij}| \leq \frac{1}{\omega_o^{n+1}} \quad (19)$$

for all $t \geq T_1, i, j = 1, 2, \dots, n+1$. Hence

$$|[e^{\omega_o A t}B]_i| \leq \frac{1}{\omega_o^{n+1}} \quad (20)$$

for all $t \geq T_1, i = 1, 2, \dots, n+1$. Note that T_1 depends on

$\omega_o A$. Let $A^{-1} = \begin{bmatrix} s_{11} & \dots & s_{1,n+1} \\ \vdots & \ddots & \vdots \\ s_{n+1,1} & \dots & s_{n+1,n+1} \end{bmatrix}$ and $e^{\omega_o A t} = \begin{bmatrix} d_{11} & \dots & d_{1,n+1} \\ \vdots & \ddots & \vdots \\ d_{n+1,1} & \dots & d_{n+1,n+1} \end{bmatrix}$. One has

$$|(A^{-1}e^{\omega_o A t}B)_i| \leq \frac{\mu}{\omega_o^{n+1}} \quad (21)$$

for all $t \geq T_1, i = 1, 2, \dots, n+1$, where $\mu = \max_{i=2, \dots, n+1} \left\{ \frac{1}{\alpha_{n+1}}, 1 + \frac{\alpha_{i-1}}{\alpha_{n+1}} \right\}$. From (17), (18), and (21), we obtain

$$|p_i(t)| \leq \frac{\delta \nu}{\omega_o^{n+1}} + \frac{\delta \mu}{\omega_o^{2n+2}} \quad (22)$$

for all $t \geq T_1, i = 1, 2, \dots, n+1$. Let $\varepsilon_{sum}(0) = |\varepsilon_1(0)| + |\varepsilon_2(0)| + \dots + |\varepsilon_{n+1}(0)|$. It follows that

$$|[e^{\omega_o A t} \varepsilon(0)]_i| \leq \frac{\varepsilon_{sum}(0)}{\omega_o^{n+1}} \quad (23)$$

for all $t \geq T_1, i = 1, 2, \dots, n+1$. From (15), one has

$$|\varepsilon_i(t)| \leq |[e^{\omega_o A t} \varepsilon(0)]_i| + |p_i(t)|. \quad (24)$$

Let $\tilde{x}_{sum}(0) = |\tilde{x}_1(0)| + |\tilde{x}_2(0)| + \dots + |\tilde{x}_{n+1}(0)|$. According to $\varepsilon_i = \frac{\tilde{x}_i}{\omega_o^{i-1}}$ and Equations (22)-(24), we have

$$\begin{aligned} |\tilde{x}_i(t)| &\leq \left| \frac{\tilde{x}_{sum}(0)}{\omega_o^{n+1}} \right| + \frac{\delta \nu}{\omega_o^{n-i+2}} + \frac{\delta \mu}{\omega_o^{2n-i+3}} \\ &= \sigma_i \end{aligned} \quad (25)$$

for all $t \geq T_1, i = 1, 2, \dots, n+1$. Q.E.D.

In summary, it has been proven that 1) when the plant model is given, the dynamic system describing the estimation error of the LESO (3) is asymptotically stable; and 2) in the absence of such model, the estimation error of the LESO (12) is bounded and its upper bound monotonously decreases with the observer bandwidth, as shown in (25). The stability of LADRC, where LESO is employed, is analyzed next.

III. STABILITY ANALYSIS OF LADRC

Assume that the control design objective is to make the output of the plant in (1) follow a given, bounded, reference signal r , whose derivatives, $\dot{r}, \ddot{r}, \dots, r^{(n)}$, are also bounded. Let $[r_1, r_2, \dots, r_n, r_{n+1}]^T = [r, \dot{r}, \dots, \dot{r}_{n-1}, \dot{r}_n]^T$. Employing the LESO of (2) in the form of (3) or (12), the ADRC control law is given as

$$u = [k_1(r_1 - \hat{x}_1) + k_2(r_2 - \hat{x}_2) + \dots + k_n(r_n - \hat{x}_n) - \hat{x}_{n+1} + r_{n+1}] / b \quad (26)$$

where $k_i, i = 1, 2, \dots, n$, are the controller gain parameters selected to make $s^n + k_n s^{n-1} + \dots + k_1$ Hurwitz. The closed-loop system becomes

$$y^{(n)}(t) = (f - \hat{x}_{n+1}) + k_1(r_1 - \hat{x}_1) + k_2(r_2 - \hat{x}_2) + \dots + k_n(r_n - \hat{x}_n) + r_{n+1}. \quad (27)$$

Note that with a well-designed ESO, the first term in the right hand side (RHS) of (27) is negligible and the rest of the terms in the RHS of (27) constitutes a generalized PD controller with a feedforward term. It generally works very well in applications but the issues to be addressed are: 1) the stability of the closed-loop system (27); and 2) the bound of the tracking error. Note that the separation principle does not apply here because of the first term in the RHS of (27).

A. Convergence of the LADRC with the Given Model of the Plant

Consider

$$\dot{\eta}(t) = N\eta(t) + g(t), \quad (28)$$

where $\eta(t) = [\eta_1(t), \eta_2(t), \dots, \eta_n(t)]^T \in \mathbb{R}^n, g(t) = [g_1(t), g_2(t), \dots, g_n(t)]^T \in \mathbb{R}^n$, and N is an $n \times n$ matrix.

Lemma 1: If N is Hurwitz and $\lim_{t \rightarrow \infty} \|g(t)\| = 0$, then $\lim_{t \rightarrow \infty} \|\eta(t)\| = 0$.

Proof. In (28), since $\lim_{t \rightarrow \infty} \|g(t)\| = 0$, then for any $\phi > 0$, there is a finite time $T_2 > 0$ such that $\|g(t)\| \leq \phi$ for all $t \geq T_2$. The response of (28) can be written as

$$\eta(t) = e^{Nt}\eta(0) + \int_0^t e^{N(t-\tau)}g(\tau) d\tau. \quad (29)$$

When $t \geq T_2$, we have

$$\begin{aligned} \|\eta(t)\| &\leq \|e^{Nt}\eta(0)\| + \|e^{Nt}\| \left\| \int_0^{T_2} e^{-N\tau}g(\tau) d\tau \right\| \\ &\quad + \int_{T_2}^t \|e^{N(t-\tau)}\| \phi d\tau. \end{aligned} \quad (30)$$

Now consider the third term of right hand side of (30). For N , there is nonsingular matrix J and block diagonal matrix $\Lambda = \text{block diag}\{\Lambda_1, \dots, \Lambda_m\}$ such that

$$N = J\Lambda J^{-1} \quad (31)$$

and each Λ_i has a single eigenvalue λ_i with its algebraic multiplicities being q_i . Suppose $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. Let $q = \max\{q_1, q_2, \dots, q_m\}$. Let us choose $\|\cdot\|_1$ or $\|\cdot\|_\infty$ for the matrix norm. It follows that

$$\|e^{\Lambda(t-\tau)}\| \leq e^{\lambda_1(t-\tau)} \sum_{k=0}^{q-1} c_k (t-\tau)^k, \quad \forall t \geq \tau, \quad (32)$$

where c_k are positive constants. Note that

$$\|e^{N(t-\tau)}\| \leq \|J\| \|e^{\Lambda(t-\tau)}\| \|J^{-1}\|. \quad (33)$$

Hence we have

$$\begin{aligned} \|\eta(t)\| &\leq \|e^{Nt}\eta(0)\| + \|e^{Nt}\| \left\| \int_0^{T_2} e^{-N\tau}g(\tau) d\tau \right\| \\ &\quad + \phi \|J\| \|J^{-1}\| \sum_{k=0}^{q-1} c_k \int_{T_2}^t e^{\lambda_1(t-\tau)} (t-\tau)^k d\tau \\ &= \|e^{Nt}\eta(0)\| + \|e^{Nt}\| \left\| \int_0^{T_2} e^{-N\tau}g(\tau) d\tau \right\| \\ &\quad + e^{\lambda_1(t-T_2)} \phi \|J\| \|J^{-1}\| \sum_{k=0}^{q-1} c_k \frac{1}{\lambda_1^{k+1}} \\ &\quad \cdot \sum_{j=0}^k (-1)^j \frac{k!}{(k-j)!} [\lambda_1(t-T_2)]^{k-j} \\ &\quad + \phi \|J\| \|J^{-1}\| \sum_{k=0}^{q-1} c_k \frac{(-1)^k k!}{\lambda_1^{k+1}}. \end{aligned} \quad (34)$$

From (34), it can be seen that

$$\begin{aligned} \lim_{t \rightarrow \infty} \|e^{Nt}\eta(0)\| &= 0 \\ \lim_{t \rightarrow \infty} \|e^{Nt}\| \left\| \int_0^{T_2} e^{-N\tau}g(\tau) d\tau \right\| &= 0 \\ \lim_{t \rightarrow \infty} \left\{ e^{\lambda_1(t-T_2)} \phi \|J\| \|J^{-1}\| \sum_{k=0}^{q-1} c_k \frac{1}{\lambda_1^{k+1}} \right. \\ &\quad \left. \cdot \sum_{j=0}^k (-1)^j \frac{k!}{(k-j)!} [\lambda_1(t-T_2)]^{k-j} \right\} &= 0. \end{aligned} \quad (35)$$

Therefore there exists $T_3 > T_2$ such that

$$\begin{aligned} \|e^{Nt}\eta(0)\| &\leq \phi, \quad \forall t \geq T_3, \\ \|e^{Nt}\| \left\| \int_0^{T_2} e^{-N\tau}g(\tau) d\tau \right\| &\leq \phi, \quad \forall t \geq T_3, \\ e^{\lambda_1(t-T_2)} \phi \|J\| \|J^{-1}\| \left\{ \sum_{k=0}^{q-1} c_k \frac{1}{\lambda_1^{k+1}} \right. \\ &\quad \left. \cdot \sum_{j=0}^k (-1)^j \frac{k!}{(k-j)!} [\lambda_1(t-T_2)]^{k-j} \right\} &\leq \phi, \quad \forall t \geq T_3. \end{aligned} \quad (36)$$

Let $c'' = \|J\| \|J^{-1}\| \sum_{k=0}^{q-1} c_k \frac{(-1)^k k!}{\lambda_1^{k+1}}$. Then we have

$$\|\eta(t)\| \leq (c'' + 3)\phi, \forall t \geq T_3. \quad (37)$$

Since ϕ can be arbitrarily small, it can be concluded that $\lim_{t \rightarrow \infty} \|\eta(t)\| = 0$.

Theorem 3: Assuming $h(x, w)$ is globally Lipschitz with respect to x , there exist constants $\omega_o > 0$ and $\omega_c > 0$, such that the closed-loop system (27) is asymptotically stable.

Proof. Define $e_i = r_i - x_i, i = 1, 2, \dots, n$. From (26), one has

$$u = [k_1(e_1 + \tilde{x}_1) + \dots + k_n(e_n + \tilde{x}_n) - (x_{n+1} - \tilde{x}_{n+1}) + r_{n+1}] / b. \quad (38)$$

It follows that

$$\begin{aligned} \dot{e}_1 &= \dot{r}_1 - \dot{x}_1 = r_2 - x_2 = e_2, \\ &\vdots \\ \dot{e}_{n-1} &= \dot{r}_{n-1} - \dot{x}_{n-1} = r_n - x_n = e_n, \\ \dot{e}_n &= \dot{r}_n - \dot{x}_n = r_{n+1} - (x_{n+1} + bu) \\ &= -k_1(e_1 + \tilde{x}_1) - \dots \\ &\quad -k_n(e_n + \tilde{x}_n) - \tilde{x}_{n+1}. \end{aligned} \quad (39)$$

Let $e = [e_1, e_2, \dots, e_n]^T \in \mathbb{R}^n, \tilde{x} = [\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n+1}]^T \in \mathbb{R}^{n+1}$, then

$$\dot{e}(t) = A_e e(t) + A_{\tilde{x}} \tilde{x}(t) \quad (40)$$

$$\text{where } A_e = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -k_1 & -k_2 & \dots & -k_{n-1} & -k_n \end{bmatrix} \text{ and}$$

$$A_{\tilde{x}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -k_1 & -k_2 & \dots & -k_n & -1 \end{bmatrix}.$$

Since $k_i, i = 1, 2, \dots, n$, are selected such that the characteristic polynomial $s^n + k_n s^{n-1} + \dots + k_1$ is Hurwitz, A_e is Hurwitz. For tuning simplicity, we just let $s^n + k_n s^{n-1} + \dots + k_1 = (s + \omega_c)^n$ where $\omega_c > 0$ and $k_i = \frac{n!}{(i-1)!(n+1-i)!} \omega_c^{n+1-i}, i = 1, 2, \dots, n$. This makes ω_c , which is the controller bandwidth, the only tuning parameter to be adjusted for the controller.

From Theorem 1, $\lim_{t \rightarrow \infty} \|A_{\tilde{x}} \tilde{x}(t)\| = 0$ if $h(x, w)$ is globally Lipschitz with respect to x . Since A_e is Hurwitz, according to Theorem 1 and Lemma 1, it can be concluded that: assuming $h(x, w)$ is globally Lipschitz with respect to x , there exist constants $\omega_o > 0$ and $\omega_c > 0$, such that $\lim_{t \rightarrow \infty} e_i(t) = 0, i = 1, 2, \dots, n$. Q.E.D.

B. Convergence of the LADRC with Plant Dynamics Largely Unknown

Now we consider the case where the plant dynamics is unknown and the LESO in the form of (12) is used instead.

Theorem 4: Assuming $h(x, w)$ is bounded, there exist a constant $\rho_i > 0$ and a finite time $T_5 > 0$ such that $|e_i(t)| \leq \rho_i, i = 1, 2, \dots, n, \forall t \geq T_5 > 0, \omega_o > 0$, and $\omega_c > 0$. Furthermore, $\rho_i = O\left(\frac{1}{\omega_c^j}\right)$ for some positive integer j .

Proof. Solving (40), we have

$$e(t) = e^{A_e t} e(0) + \int_0^t e^{A_e(t-\tau)} A_{\tilde{x}} \tilde{x}(\tau) d\tau. \quad (41)$$

According to (40) and Theorem 2, one has

$$\begin{aligned} [A_{\tilde{x}} \tilde{x}(\tau)]_{i=1, \dots, n-1} &= 0 \\ |[A_{\tilde{x}} \tilde{x}(\tau)]_n| &\leq k_s \sigma_i = \gamma, \forall t \geq T_1 \end{aligned} \quad (42)$$

where $k_s = 1 + \sum_{i=1}^n k_i$. Similar to Theorem 3, choose $k_i = \frac{n!}{(i-1)!(n+1-i)!} \omega_c^{n+1-i}, i = 1, 2, \dots, n$, such that A_e is Hurwitz. Define $\Psi = [0 \ 0 \ \dots \ 0 \ \gamma]^T$. Let $\varphi(t) = \int_0^t e^{A_e(t-\tau)} A_{\tilde{x}} \tilde{x}(\tau) d\tau$. It follows that

$$|\varphi_i(t)| \leq |(A_e^{-1} \Psi)_i| + |(A_e^{-1} e^{A_e t} \Psi)_i|. \quad (43)$$

and

$$\begin{aligned} |(A_e^{-1} \Psi)_1| &= \frac{\gamma}{k_1} = \frac{\gamma}{\omega_c^n} \\ |(A_e^{-1} \Psi)_i|_{i=2, \dots, n} &= 0 \end{aligned} \quad (44)$$

Since A_e is Hurwitz, there exists a finite time $T_4 > 0$ such that

$$| [e^{A_e t}]_{ij} | \leq \frac{1}{\omega_c^{n+1}} \quad (45)$$

for all $t \geq T_4, i, j = 1, 2, \dots, n$. Note that T_4 depends on A_e . Let $T_5 = \max\{T_1, T_4\}$. It follows that

$$| (e^{A_e t} \Psi)_i | \leq \frac{\gamma}{\omega_c^{n+1}} \quad (46)$$

for all $t \geq T_5, i = 1, 2, \dots, n$, and

$$| (A_e^{-1} e^{A_e t} \Psi)_i | \leq \begin{cases} 1 + \sum_{i=2}^n k_i \frac{\gamma}{\omega_c^n \omega_c^{n+1}} \\ \frac{\gamma}{\omega_c^{n+1}} \end{cases} \Big|_{i=1} \quad (47)$$

for all $t \geq T_5$. From (43), (44), and (47), we obtain

$$|\varphi_i(t)| \leq \begin{cases} \frac{\gamma}{\omega_c^n} + \frac{1 + \sum_{i=2}^n k_i \frac{\gamma}{\omega_c^n \omega_c^{n+1}}}{\omega_c^n} \\ \frac{\gamma}{\omega_c^{n+1}} \end{cases} \Big|_{i=1} \quad (48)$$

for all $t \geq T_5$. Let $e^{A_e t} = \begin{bmatrix} o_{11} & \dots & o_{1n} \\ \vdots & \ddots & \vdots \\ o_{n1} & \dots & o_{nn} \end{bmatrix}$ and $e_s(0) = |e_1(0)| + |e_2(0)| + \dots + |e_n(0)|$. It follows that

$$| [e^{A_e t} e(0)]_i | \leq \frac{e_s(0)}{\omega_c^{n+1}} \quad (49)$$

for all $t \geq T_5$, $i = 1, 2, \dots, n$. From (41), one has

$$|e_i(t)| \leq \left| [e^{A_c t} e(0)]_i \right| + |\varphi_i(t)|. \quad (50)$$

According to (42), (48)-(50), we have

$$|e_i(t)| \leq \begin{cases} \left| \frac{e_s(0)}{\omega_c^{n+1}} + \frac{k_s \sigma_i}{\omega_c^n} + \frac{\left(1 + \sum_{i=2}^n k_i\right) k_s \sigma_i}{\omega_c^{2n+1}} \right|_{i=1} \\ \left| \frac{e_s(0) + k_s \sigma_i}{\omega_c^{n+1}} \right|_{i=2, \dots, n} \end{cases} \leq \rho \quad (51)$$

for all $t \geq T_5$, $i = 1, 2, \dots, n$, where $\rho_i = \max \left\{ \frac{e_s(0)}{\omega_c^{n+1}} + \frac{k_s \sigma_i}{\omega_c^n} + \frac{\left(1 + \sum_{i=2}^n k_i\right) k_s \sigma_i}{\omega_c^{2n+1}}, \frac{e_s(0) + k_s \sigma_i}{\omega_c^{n+1}} \right\}$. Q.E.D.

In summary, it has been shown that 1) with the given model of the plant, the closed-loop system (27) is asymptotically stable; and 2) with plant dynamics largely unknown, the tracking error and its up to $(n-1)^{th}$ order derivatives of LADRC are bounded and their upper bounds monotonously decrease with the controller bandwidth, as shown in (51).

IV. CONCLUDING REMARKS

Existing estimators and their characteristics are first summarized in this paper, concerning unknown disturbance and plant dynamics. The main result in this paper is the analysis of the stability and tracking characteristics of a particular class of such observers, LESO, and the associated feedback control system, LADRC. Both design scenarios, with and without a detailed mathematical model of the plant, are considered. It is shown that the asymptotic stability is assured in the former and boundedness of the estimation error and the closed-loop tracking error in the later. Furthermore, it is shown that the tracking error monotonously decreases with the control loop bandwidth.

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