On the necessity, scheme and basis of the linear-nonlinear switching in active disturbance rejection control

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On the Necessity, Scheme and Basis of the Linear-Nonlinear Switching in Active Disturbance Rejection Control

Jie Li, Yuanqing Xia, Senior Member, IEEE, Xiaohui Qi, Zhiqiang Gao, Member, IEEE

Abstract—Active disturbance rejection control (ADRC) was originally given in a nonlinear gain structure to better accommodate the dynamic uncertainties and disturbances. However, the resulting complexity in theoretical analysis and in parameter tuning inhibits the applications of ADRC. ADRC was linearized and parameterized for practical convenience. Since linear ADRC (LADRC) and nonlinear ADRC (NLADRC) each has its own advantages and disadvantages, choosing between LADRC and NLADRC is rather difficult. As a matter of fact, there is a lack of quantitative analysis and comparison between LADRC and NLADRC. This paper firstly gives an easy solution in the parameter tuning of the nonlinear extended state observer (ESO), followed by a quantitative analysis and comparison study on LADRC and NLADRC; then a LADRC/NLADRC switching control (SADRC) scheme is proposed and its stability is analyzed; lastly, the SADRC scheme is verified by experiment using the ball-beam platform. The proposed SADRC takes the advantage of the additional performance improvement associated with the NLADRC but make it easier to use.

Index Terms—Active disturbance rejection control, extended state observer, switching control, parameter tuning, stability analysis.

I. INTRODUCTION

In the domain of disturbance/uncertainty estimation and attenuation techniques, linear and nonlinear methods/forms are both widely existed [1]. ADRC, as a practical and popular control technique, was firstly developed as nonlinear structure by Han [2] and latter linearized and parameterized by Gao [3], a cooperator of Han. The NLADRC prefers to use nonlinear functions in the design of the observer and the control law, which is potentially much more effective in tolerance to uncertainties and disturbances and improvement of system dynamics. However, stability and performance analysis are very difficult for such nonlinear systems, and too many parameters make the tuning rather difficult, which inhibit the application of ADRC. The LADRC is superior to the NLADRC in parameter tuning and theoretical analysis. Since the LADRC was proposed, it greatly promotes the theoretical analysis and application of ADRC. Since LADRC and NLADRC each has its own advantages and disadvantages, choosing between LADRC and NLADRC is rather difficult. To this end, on one hand, theoretical study about NLADRC has never been stagnant and there is endeavor to make it more easy to be used; on the other hand, there are attempts to further improve the performance of LADRC. However, there are still lack of quantitative analysis and comparison between LADRC and NLADRC, and there are some ambiguities in our understanding about them. Therefore, in this paper, a comparison research on LADRC and NLADRC will be carried out and a SADRC scheme will be further presented to fully take advantages of them. At the same time, parameter tuning and stability analysis are two important and unavoidable problems for ADRC, for which we will present a literature review and give our solutions.

Parameter tuning is always a very hot topic for ADRC, and previous studies can also be classified into two groups, i.e., one for LADRC and another for NLADRC. Parameter tuning methods for LADRC can be further classified into constant gain methods and adaptive variable gain methods. Constant gain methods are mainly the “bandwidth” method [3] and its modified versions [4]. Although these constant gain methods may not be optimal, they are very popular for convenience and effectiveness, which is crucial for practical use. The adaptive variable gain methods [5], [6] are recently proposed to further improve the performance limited by constant gain. These methods are effective and can provide solutions for some complex situations, but they require much knowledge of plant and the process is much more complex.

Parameter tuning methods for NLADRC can be further classified into empirical formula-based methods, artificial intelligence-based approaches and other methods. Han proposed empirical formula-based methods [2], which are based on sample time or fibonacci sequence. Though these methods are very simple and effective in simulation, they may be limited in practical application. That is because physical limitation, such as noise and bandwidth, are rarely taken into account. In fact, parameter tuning is a trade-off between performance and physical limitations. Artificial intelligence-based methods, such as neural network [7], chaos particle swarm optimization [8], continuous action reinforcement learning automata [9], are used to optimize the parameters. Although these approaches seem to guarantee a good performance, they are difficult to be used and not general for practical application. Besides, other methods like parameter tuning based on time scale [10], time-
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...drawn in Section V.

...are shown in Section IV. Finally, some concluding remarks are

...a detailed SADRC scheme is proposed, followed by stability

...method for the NLESO is proposed; a quantitative comparison

...the algorithm of ADRC is simply introduced; parameter tuning

...summarized as below:

...practical application, this paper will present some further and

...NLESO only considers certain special cases; (3) the stability

...on LADRC and NLADRC and their own characteristics are

...ing aspects: (1) there is lack of quantitative comparison study

...previous studies can be mainly classified into four groups,

...In fact, previous work [25] has proposed a SADRC scheme,

...Signal. In this paper, we will transform the SADRC-based

...in this paper it is also

...where

\[ u = \frac{u_0 - z_{n+1}}{b_0} \]  

where one of the form of \( u_0 \) may employ the following combination

\[ u_0 = \sum_{i=1}^{n} k_i \text{fal}(v_i - z_i, \alpha_i, \delta_i) \]  

where \( k_i (i = 1, 2, \ldots, n) \) are the controller gains, and when

...A. ADRC Algorithm

ADRC generally consists of generating transition process (TP), ESO and control law (CL). Generating TP is a very useful way to solve the contradictory between overshoot and quickness. Usually, we make use of TP as well as its n-order derivatives to construct the control law. There are many ways to generate TP, and the general form is as follows

\[
\begin{align*}
\dot{v}_1 &= v_2 \\
\dot{v}_2 &= v_3 \\
&\vdots \\
\dot{v}_{n-1} &= v_n \\
\dot{v}_n &= \lambda^n \psi(v_1 - r, v_2, \ldots, v_n)
\end{align*}
\]  

where \( r \) is the input force, \( v_i (i = 1, 2, \ldots, n) \) are the outputs, \( \lambda \) is the adjustable speed factor, \( \psi(v_1 - r, v_2, \ldots, v_n) \) is a solution that guarantees the fast convergence from \( v_1 \) to \( r \).

ESO is used to estimate the system states and the total disturbance, which is the core and essence of the ADRC. Referring to [2], a general ESO is designed in the following form

\[
\begin{align*}
e &= z_1 - y \\
\dot{z}_1 &= z_2 - \beta_{01} \varphi_1(e) \\
\dot{z}_2 &= z_3 - \beta_{02} \varphi_2(e) \\
&\vdots \\
\dot{z}_n &= z_{n+1} - \beta_{0n} \varphi_n(e) + b_0 \cdot u \\
\dot{z}_{n+1} &= -\beta_{0(n+1)} \varphi_{n+1}(e)
\end{align*}
\]  

where the inputs of the ESO are the regulated output \( y \) and the control input \( u \) of the plant; the outputs of the ESO are \( z_i (i = 1, 2, \ldots, n) \) which provide an estimation of the system states and \( z_{n+1} \) which provides an estimation of the total disturbance; \( \beta_{0i} (i = 1, 2, \ldots, n + 1) \) are the observer gains, \( e \) is the observer error “\( z_1 - y \)”, and in this paper it is also denoted by \( e_1 \) as usual practice. \( \varphi_i(e) (i = 1, 2, \ldots, n + 1) \) are nonlinear functions, particularly, defined as

\[ \varphi_i(e) = \text{fal}(e, \alpha_i, \delta) = \begin{cases} 
\frac{e^{\delta_1 - \alpha_i}}{|e|^{\alpha_i}} \text{sgn}(e) & |e| \leq \delta \\
|e|^{\alpha_i} \text{sgn}(e) & |e| > \delta
\end{cases} \]  

where \( \alpha_i \) and \( \delta \) are important parameters to be predetermined. This nonlinear function plays an important role in the newly proposed NLADRC framework, due to its characteristics of “small error, big gain; big error, small gain” when \( \alpha_i < 1 \). When \( \alpha_i = 1 \), this nonlinear \( \text{fal}(e, \alpha_i, \delta_i) \) function turns into a linear one.

Control law is used to restrain the residual error and achieve the desired control goal. The control law is designed as

\[ u = \frac{u_0 - z_{n+1}}{b_0} \]  

where the rest of this paper is organized as follows. In Section II, the algorithm of ADRC is simply introduced; parameter tuning method for the NLESO is proposed; a quantitative comparison study on LADRC and NLADRC is carried out. In Section III, a detailed SADRC scheme is proposed, followed by stability analysis. The hardware validations using the ball-beam system are shown in Section IV. Finally, some concluding remarks are drawn in Section V.
B. Parameter Tuning of NLESO

As a matter of fact, we are preferable to use low order ESO like second and third order ESO in practical application to simplify the design and analysis as well as reduce phase lag and improve performance. Hence, we here mainly focus on the parameter tuning of second and third order NLESO. There are three types of parameters in the NLESO, i.e., gains $\beta_i$, power $\alpha_i$ and linear range $\delta$. For the power $\alpha_i (i = 1, 2, 3)$, a set of effective empirical values are $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.25$. Then we further consider the linear range $\delta$, and let

$$\text{fal}(\epsilon, \alpha_i, \delta) = \frac{\text{fal}(\epsilon, \alpha_i, \delta)}{\epsilon} = \lambda_0(\epsilon)$$

thus, $\text{fal}(\epsilon, \alpha_i, \delta)$ can be seen as a linear function $\epsilon$ with varying gain $\lambda_0(\epsilon)$. Take $\alpha = 0.25$, $\delta = 0.1$ for example, we can obtain the characteristics curve of the gain function $\lambda_0(\epsilon)$ shown in Fig. 1. To quickly recover the system states during the transient period and reduce the effect of measurement noise afterwards, the steady state error $e'$ should be located in the nonlinear range of the function $\lambda_0(e)$, i.e., $\delta < e'$. There exists a range when $e' \in (e_1, e_2)$ and $\delta < e_1 < e_2$, the NLESO has fast state reconstruction and good measurement noise attenuation. Therefore, due to the simple nonlinear function $\text{fal}(\epsilon, \alpha_i, \delta)$, the NLESO has the merits of the improved high-gain observers [12], [13].

![Characteristic curve of the gain function $\lambda_0(\epsilon)$](image)

Fig. 1. Characteristic curve of the gain function $\lambda_0(\epsilon)$.

Of course, the steady state error $e'$ is closely related to the gains of NLESO, which will be quantitatively illustrated latter. Here we consider the gain tuning of the NLESO. We adopt particle swarm optimization algorithm combing with empirical formula method to tune the parameters, which imitate Han’s way, i.e., optimize parameters by adopting certain optimization method through simulation for NLESO [2], and Gao’s pole placement method for LESO [3]. The performance of NLESO is mainly influenced by the amplitude and derive of disturbance, sample time (denoted by $h$), noise, which are considered in the optimization process.

Consider the following one order system

$$\begin{cases}
\dot{x} = M \text{sign}(\sin(\omega t)) \\
y = x
\end{cases}$$

(7)

For the above plant, a second order ESO is designed as follows

$$\begin{cases}
\dot{e} = z_1 - y \\
z_1 = z_2 - \beta_{01} \cdot \text{fal}(e, 1, 0.005) \\
z_2 = -\beta_{02} \cdot \text{fal}(e, 0.5, 0.005)
\end{cases}$$

(8)

The rest of optimization process is just the same with the above one order system, hence it is omitted here. The gains are optimized as shown in Table 1.

<table>
<thead>
<tr>
<th>Parameter optimization for second order NLESO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
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<tr>
<td>0.001</td>
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<tr>
<td>0.001</td>
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<td>0.01</td>
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<td>0.01</td>
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</tbody>
</table>

Through curve fitting (imitating Han’s way [2]), the gains $\beta_{01}$ and $\beta_{02}$ in Table 1 can be approximately denoted by

$$\beta_{01} = 2 \omega_o, \beta_{02} = \omega_o^2 / 3$$

(10)

where $\omega_o$ is a parameter borrowed from the bandwidth method.

Consider the following second order system

$$\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = M \text{sign}(\sin(\omega t)) \\
y = x_1
\end{cases}$$

(11)

For the above plant, a third order ESO is designed as follows

$$\begin{cases}
\dot{e} = z_1 - y \\
\dot{z}_1 = z_2 - \beta_{01} \cdot \text{fal}(e, 1, 0.005) \\
\dot{z}_2 = z_3 - \beta_{02} \cdot \text{fal}(e, 0.5, 0.005) \\
\dot{z}_3 = -\beta_{03} \cdot \text{fal}(e, 0.25, 0.005)
\end{cases}$$

(12)

Here we consider the relationship among the amplitude $M$, sample time $h$ and the parameters $\beta_{01}$ and $\beta_{02}$. The optimization adopts the following performance index [2]

$$J = \frac{1}{M} \int_0^T |x_3(t) - z_3(t)| dt$$

(9)

The first step is to use a LESO to obtain its suitable gains $\beta_{01}$ and $\beta_{02}$ by the “bandwidth” method under a certain degree noise. Since the first function $\text{fal}(e, 1, 0.005)$ in Eqs. (8) is a linear one, during the optimization process, the gain $\beta_{01}$ of NLESO remains the same, and only the gain $\beta_{02}$ needs to be optimized. The gains are optimized as shown in Table 2.

<table>
<thead>
<tr>
<th>Parameter optimization for third order NLESO</th>
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<tbody>
<tr>
<td>$h$</td>
</tr>
<tr>
<td>0.001</td>
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</table>
The gains $\beta_{01}, \beta_{02}$ and $\beta_{03}$ in Table 2 can be approximately denoted by

$$
\beta_{01} = 3\omega_o, \beta_{02} = 3\omega_o^2/5, \beta_{03} = \omega_o^3/9. \tag{13}
$$

It can be seen that the parameters of NLESO can be easily determined by empirical formulas. Since there is only one parameter $\omega_o$, it can be easily determined by try and error in real engineering practice, just like the parameter tuning of LADRC. It is worthy noting that $\omega_o$ is always a trade-off between performance and the real physical limitation (such as noise, sample time, and so on). Therefore, we don’t theoretically seek for a so-called optimization of $\omega_o$. In fact, it almost impossible to get optimal results since the real situations are complex.

**Remark 2.1** Although a specific $\delta = 0.005$ is used in the optimization, they are still useful for different $\delta$ in a certain range. At the same time, the coefficients of the formulas (10) and (13) are not strict. In formula (10), the “3” in $\beta_{02} = \omega_o^2/3$ can be replaced with a number in 3~5; in formula (13), the “5” in $\beta_{02} = 3\omega_o^2/5$ can be replaced with 3~5 and the “9” in $\beta_{03} = \omega_o^3/9$ can be replaced with 7~12. An assisted simulation can be used for a given system to determine a more suitable combination of these coefficients. In fact, the performance of ESO is not sensitive to its gains in a certain range. A large number of simulations with experiments on ball-beam system and inverted pendulum system have verified these formulas [2], [25], [26].

**C. A comparison between LADRC and NLADRC**

It is said that the ADRC is NLADRC if it adopts either or both NLESO and nonlinear control law; otherwise, it is LADRC. We will quantitatively and qualitatively illustrate the differences between them.

1) A comparison between LEO and NLESO: In general, the NLESO is superior to LEO, but the performance of the former will decrease sharply if the amplitude or derivative of the total disturbance turns too big. Consider a one order system described by

$$
\dot{x} = f(x, w(t)) \tag{14}
$$

where $f(\cdot)$ is an uncertain function, $w(t)$ is an unknown external disturbance.

Let $x_1 = x, x_2 = f(\cdot)$, then the system (14) is augmented as

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(t) \\
y &= x_1
\end{align*} \tag{15}
$$

where $-h(t) = \dot{f}(\cdot)$ is assumed unknown but bounded. Correspondingly, a second order ESO is designed, and the observer estimation error is defined as

$$
\begin{align*}
e_1 &= z_1 - x_1, e_2 = z_2 - x_2 \\
\dot{e}_1 &= e_2 - \beta_{01} e_1 \\
\dot{e}_2 &= h(t) - \beta_{02} \text{fal}(e_1, \alpha_1, \delta)
\end{align*} \tag{16}
$$

Make the following linear transformation

$$
\begin{align*}
\dot{z}_1' &= e_1 \\
\dot{z}_2' &= e_2 - \beta_{01} e_1
\end{align*} \tag{17}
$$

which is equal to the following system

$$
\begin{align*}
\dot{z}_1' &= z_2' \\
\dot{z}_2' &= -\beta_{02} \text{fal}(z_1', \alpha_1, \delta) - \beta_{01} z_2' + h(t)
\end{align*} \tag{18}
$$

Construct the following Lyapunov candidate function

$$
V = \beta_{02} \int_0^t \text{fal}(z_1', \alpha_1, \delta) \dot{z}_1' + \frac{z_2'^2}{2} \tag{19}
$$

Computing the derivative of (19) along the solution of system (18), we obtain

$$
\frac{dV}{dt} \big|_{(18)} = \beta_{02} \text{fal}(z_1', \alpha_1, \delta) z_2' + z_2' \cdot \dot{z}_2' = z_2'(h(t) - \beta_{01} z_2') \tag{20}
$$

If $|z_2'| > \frac{h_0}{\beta_{01}}$, then $\frac{dV}{dt} < 0$, i.e., the trajectories of the system will eventually go into the range $|z_2'| \leq \frac{h_0}{\beta_{01}}$. When the system (18) goes into steady state, we have

$$
\begin{align*}
z_2' &= 0 \\
\{-\beta_{02} \text{fal}(z_1', \alpha_1, \delta) - \beta_{01} z_2' + h_0 &= 0 \tag{21}
\end{align*}
$$

Let $\alpha = 0.5$, then we obtain the steady state error of the second order NLESO

$$
|e_1| \leq \left(\frac{h_0}{\beta_{02}}\right)^2, |e_2| \leq \beta_{01} \left(\frac{h_0}{\beta_{02}}\right)^2 \tag{22}
$$

Assume $h(t)$ is a constant $h_0$ and $\delta < |e_1|$, and the gains of the second order NLESO are set as (10), then the steady state error are given as follows

$$
|e_1| = \left(\frac{3h_0}{\omega_o^2}\right)^2, |e_2| = 2\omega_o \left(\frac{3h_0}{\omega_o^2}\right)^2 \tag{23}
$$

If $\alpha = 1$, then the ESO turns into a LEO. Set the gains of the LEO as $\beta'_{01} = 2\omega_o, \beta'_{02} = \omega_o^2$ (the same $\omega_o$ as that in (23)), then steady state errors are given as follows

$$
|e_1'| = \frac{h_0}{\omega_o^2}, |e_2'| = 2\omega_o \frac{h_0}{\omega_o^2} \tag{24}
$$

Compare (23) with (24), if $h_0 < \frac{\omega_o^2}{\beta'_{02}}$, then $|e_1| < |e_1'|, |e_2| < |e_2'|$; if $h_0 \geq \frac{\omega_o^2}{\beta'_{02}}$, then $|e_1| \geq |e_1'|, |e_2| \geq |e_2'|$. With the rise of the amplitude of $h_0$, the estimate errors of the NLESO will increase more quickly than the LEO due to the exponential power $0 < \alpha < 1$. At the same time, with the rise of $e_1$, the equivalent gain $\alpha_{01}(e_1)$ turns smaller, which leads to the performance of NLESO degrading sharply. Therefore, it can be concluded that the second order NLESO is not always superior to the same order LEO. Similar conclusions can be obtained for other high order ESO and they are omitted here.

2) A Comparison between Linear and Nonlinear CL: Here we only consider the control law (5), which is similar to traditional Proportional-Differential (PD) control law. When $\alpha_i = 1$, this nonlinear control law turns into a linear one. Although the nonlinear control law is effective in simulations, however, compared with the linear one, there are several disadvantages: its parameter tuning is complex; the control signal is not as smooth as the linear one, which causes adverse effects to the actuator; it amplifies the noise during the steady state due to high gains; it has an adverse impact on the stability. Hence, we adopt the linear control law in this research.
3) An overall evaluation about LADRC and NLADRC: According to the above analysis and common sense, the characteristics of LADRC and NLADRC are systematically summarized respectively as shown in Table 3.

<table>
<thead>
<tr>
<th>TABLE III</th>
<th>CHARACTERISTICS OF LADRC AND NLADRC</th>
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<tbody>
<tr>
<td><strong>Advantages:</strong></td>
<td></td>
</tr>
<tr>
<td>1. It is convenient to carry on the theory research;</td>
<td></td>
</tr>
<tr>
<td>2. The performance will not significantly deteriorate when the amplitude or derivative of the total disturbance turns big.</td>
<td></td>
</tr>
<tr>
<td><strong>LADRC Disadvantages:</strong></td>
<td></td>
</tr>
<tr>
<td>1. It may be less effective than the NLADRC;</td>
<td></td>
</tr>
<tr>
<td>2. The initial state errors may lead to “peaking phenomenon” due to constant gain;</td>
<td></td>
</tr>
<tr>
<td><strong>Advantages:</strong></td>
<td></td>
</tr>
<tr>
<td>1. It may be more effective than the LADRC, such as higher tracking precision, stronger anti-interference capability, quicker response, and so on;</td>
<td></td>
</tr>
<tr>
<td>2. It is relatively insensitive to initial state errors due to the nonlinear mechanism – “big error, small gains”.</td>
<td></td>
</tr>
<tr>
<td><strong>NLADRC Disadvantages:</strong></td>
<td></td>
</tr>
<tr>
<td>1. It is difficult to analyze stability and performance, and the traditional frequency theory is difficult to be applied to;</td>
<td></td>
</tr>
<tr>
<td>2. The performance will sharply degrade when the amplitude or derivative of the total disturbance turns big to a certain degree;</td>
<td></td>
</tr>
</tbody>
</table>

III. A SCHEME OF SADRC

Since both the LADRC and NLADRC have their own advantages and disadvantages, it will be very nice if the merits of LADRC and NLADRC can be effectively integrated. To achieve this goal, a scheme of SADRC is proposed in this section.

A. A Scheme of SADRC

Since we adopt the linear control law in NLADRC in this research, the essence of SADRC is a switch between LESO and NLESO. A scheme of SADRC is given as follows:

(1) Initial operation phase: if there are possible initial state errors between the states of the plant and the ESO, NLESO is adopted during a transition time T (artificially set) to avoid “peaking phenomenon”; otherwise, this step is omitted and directly goes into the following step;

(2) Normal operation phase: after step (1), the controller automatically switches between LESO and NLESO according to the error $e$. Choose a specific $\delta' > \delta$, when the error $e < \delta'$, NLESO starts to work; otherwise, LESO begins to work. This $\delta'$ is a general boundary of the performance of LESO and NLESO, i.e., when $e < \delta'$ NLESO is superior to LESO, whereas LESO is superior to NLESO. Latter we will illustrate how to determine the $\delta'$.

The above process is also shown in Fig. 2. In general, this scheme of SADRC synthesizes both the merits of LADRC and NLADRC. Besides, the frequency domain methods can still be used for SADRC to estimate the usually used indexes, such as stability margin, since LADRC generally works in the worse conditions.

This single parameter “$\delta'$” can be easily determined by simulation, experiment or theoretical compute. Add a big disturbance in the nominal model during the simulation or on the real plant during the experiment, then repeatedly tune the “$\delta'$” until the performance is superior to both NLADRC and LADRC. Also, we can compute “$\delta'$” from theoretical viewpoint. In fact, for the aforementioned second order ESO, when both the $\omega_o$ of the NLESO and $\omega'_o$ of the LESO are the same, then $|e_1| = |e'_1|, |e_2| = |e'_2|$ when $h_0 = \frac{\omega^2_o}{\beta}$ Thus, $\delta'$ can be determined as $\delta' = \frac{h_0}{\beta} = \frac{\omega^2_o}{\beta}$, which is the boundary of the performance of LESO and NLESO.

For general situations, we take a third order NLESO for example to determine the “$\delta'$” by compute. For a third order ESO, the estimate error equations are given as follows [2]

$$
\begin{align*}
\dot{e}_1 &= e_2 - \beta_{01} e_1 \\
\dot{e}_2 &= e_3 - \beta_{02} f(\alpha_1, e_1, \delta) \\
\dot{e}_3 &= h_0 - \beta_{03} f(\epsilon, e_1, \epsilon_2) 
\end{align*}
$$

(25)

When the system (25) goes into steady state, the steady state errors for the third order NLESO are given as follows

$$
|e_1| = \left(\frac{h_0}{\beta_{03}}\right)^4, |e_2| = \beta_{01} \left(\frac{h_0}{\beta_{03}}\right)^4, |e_3| = \beta_{02} \left(\frac{h_0}{\beta_{03}}\right)^2
$$

(26)

Correspondingly, the steady state errors for the third order LESO are given as follows

$$
|e'_1| = \frac{h_0}{\beta'_{03}}, |e'_2| = \beta'_{01} \left(\frac{h_0}{\beta'_{03}}\right)^4, |e'_3| = \beta'_{02} \left(\frac{h_0}{\beta'_{03}}\right)^2
$$

(27)

Assuming

$$
\beta_{01} = 3\omega_o, \beta_{02} = 3\omega_o^2/5, \beta_{03} = \omega_o^3/9.
$$

(28)

$$
\beta'_{01} = 3\omega'_o, \beta'_{02} = 3\omega'_o^2/5, \beta'_{03} = \omega'_o^3/9.
$$

(29)

Let $\omega'_o = 2\omega_o$, then

$$
\beta'_{01} = 6\omega_o, \beta'_{02} = 12\omega_o^2, \beta'_{03} = 8\omega_o^3.
$$

(30)
Substitute Eqs. (28) and (30) into Eqs. (26) and (27), and compare \(|e_1|, |e_2|, |e_3|\) with \(|e'_1|, |e'_2|, |e'_3|\), respectively, then we can obtain the following critical conditions.

\[
\begin{align*}
|e_1| &= |e'_1|, \quad h_0 = \omega_0^3/37 \\
|e_2| &= |e'_2|, \quad h_0 = \omega_0^3/30 \\
|e_3| &= |e'_3|, \quad h_0 = \omega_0^3/32
\end{align*}
\] (31)

Let \(h'_0 \in [\omega_0^3/37, \omega_0^3/30]\), and to make the best of nonlinear mechanism in the whole, the \(\delta'\) is set as

\[
\delta' = \left(\frac{h'_0}{\omega_0^3/9}\right)^4 \in [0.003, 0.008]
\] (32)

According to (32), \(\delta'\) can be further tuned and determined by experiments.

**B. Stability analysis**

Stability is a basic requirement that any control system must meet. We study the stability of SADRC-based nominal control system, and consider the following typical linear single-input single-output (SISO) system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots &= \\
\dot{x}_n &= a_n x_1 + a_{n-1} x_2 + \cdots + a_1 x_n + b_0 u \\
y &= x_1
\end{align*}
\] (33)

where \(y\) is the regulated output, \(u\) is the control input, \(b_0\) is the gain coefficient of the control input, \(x_i (i = 1, 2, \ldots, n)\) are the system states and \(a_i (i = 1, 2, \ldots, n)\) are their corresponding gain coefficients.

Assume that the LESO’s gains \(\beta'_0 (i = 1, 2, \ldots, n + 1)\) are \(\lambda_i\) multiple of the NLESO’s gains \(\beta_n (i = 1, 2, \ldots, n + 1)\), then the switch form of ESO can be uniformly denoted as follows:

\[
\begin{align*}
e &= z_1 - y \\
\dot{z}_1 &= z_2 - \beta_0 \cdot \phi_1 (e) \\
\dot{z}_2 &= z_3 - \beta_0 \cdot \phi_2 (e) \\
\vdots &= \\
\dot{z}_n &= z_{n+1} - \beta_0 \cdot \phi_n (e) + b_0 \cdot u \\
\dot{z}_{n+1} &= -\beta_0 (\phi_n (e) + \lambda_i \cdot \phi_n (e)) + b_0 \cdot u
\end{align*}
\] (34)

where

\[
\phi_i (e) = \begin{cases} 
\varphi_i (e) & |e| \leq \delta' \\
\lambda_i \cdot e & |e| > \delta'
\end{cases}
\] (35)

The control law is assumed to be linear, i.e., \(\alpha'_i = 1 (i = 1, 2, \ldots, n)\). Let the input \(r\) be zero, and all the outputs \(v_i (i = 1, 2, \ldots, n)\) of TP be zeros. Substitute Eqs. (4) and (5) into Eq. (33), and let \(X = [x_1, x_2, \ldots, x_n]^T, Z = [z_1, z_2, \ldots, z_{n+1}]^T\), and then we obtain

\[
\begin{align*}
\dot{X} &= A_{11} X + A_{12} Z \\
y &= x_1
\end{align*}
\] (36)

where

\[
A_{11} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_n \\
an & a_{n-1} & \cdots & a_2 & a_1
\end{bmatrix} \in \mathbb{R}^{n \times n},
\]

\[
A_{12} = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
-k_1 & \cdots & -k_n & -1
\end{bmatrix} \in \mathbb{R}^{n \times (n+1)}.
\]

Substitute Eqs. (4) and (5) into Eq. (34), and we have

\[
\dot{Z} = A_{21} Z + A_{22} \dot{u}
\] (37)

where

\[
\dot{u} = -[\phi_1 (e) \cdots \phi_{n+1} (e)]^T,
\]

\[
A_{21} = \begin{bmatrix}
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-k_1 & -k_2 & \cdots & -k_n & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\beta_1 & 0 & \cdots & 0 & 0 \\
\beta_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \beta_{n+1}
\end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)},
\]

\[
A_{22} = \begin{bmatrix}
0 & 0 & \cdots & 0 & \beta_{n+1}
\end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.
\]

Combine Eqs. (36) and (37), and we have

\[
\begin{align*}
\dot{X} &= A_{11} X + A_{12} Z \\
\dot{Z} &= A_{21} Z + A_{22} \dot{u} \\
e &= c_1^T X + c_2^T Z
\end{align*}
\] (38)

where \(c_1 = [-1, 0, \ldots, 0]^T \in \mathbb{R}^n, c_2 = [1, 0, \ldots, 0]^T \in \mathbb{R}^n\). Then we have

\[
\dot{x} = A \ddot{x} + B u
\]

\[
e = c^T \ddot{x}
\] (39)

where

\[
\dot{x} = [X, Z]^T \in \mathbb{R}^{2n}, \quad A = \begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{21}
\end{bmatrix} \in \mathbb{R}^{(2n+1) \times (2n+1)}.
\]

\[
B = \begin{bmatrix}
0 \\
A_{22}
\end{bmatrix} \in \mathbb{R}^{(2n+1) \times (n+1)}.
\]

\[
c^T = [c_1^T, c_2^T] \in \mathbb{R}^{2n+1}.
\]

This is a multiple-input-single-output (MISO) Lurie-like system, which can be further extended into a MIMO Lurie system as follows

\[
\dot{x} = \ddot{A} \ddot{x} + B \ddot{u}
\]

\[
\sigma = C^T \ddot{x}
\] (40)

where

\[
\sigma = [e, 0 \cdots 0]^T \in \mathbb{R}^{n+1},
\]

\[
C^T = [c_1^T, \ldots, c^n^T, c^T]^T \in \mathbb{R}^{(n+1) \times (2n+1)}.
\]

which can be further transformed into the following equations.

\[
\dot{x} = \ddot{A} \ddot{x} + B \ddot{u}
\]

\[
\sigma = C^T \ddot{x}
\] (41)

where

\[
\ddot{u} = -[\phi_1 (e) - \mu_1 e \cdots \phi_{n+1} (e) - \mu_{n+1} e]^T \in \mathbb{R}^{n+1},
\]

\[
\zeta = [\mu_1 \ldots \mu_{n+1}]^T \in \mathbb{R}^{n+1}, \quad A = \ddot{A} - B \zeta C^T.
\]

\[
\mu_i = \min\{\lambda_i, \delta_i, -1, \delta_i^{-1}\}.
\]

Let \(\ddot{u} = -[\phi_1 (e) \cdots \phi_{n+1} (e)]^T \in \mathbb{R}^{n+1}\), where \(\phi_i (e) = \phi_i (e) - \mu_i e, i = 1, 2, \ldots, n + 1\).
Eqs. (41) is a MIMO Lurie-like system, for which absolute stability is often preferentially considered. Relevant notations about “absolute stability” are firstly introduced for the subsequent work.

**Definition 3.1** The memoryless time-invariant nonlinear function \( \varphi : \mathbb{R} \to \mathbb{R} \) is said to belong to the sector \([\kappa_1, \kappa_2]\) \((0 \leq \kappa_1 < \kappa_2 \leq \infty)\), denoted as \( \varphi \in F[\kappa_1, \kappa_2] \), if the following inequality holds:

\[
\varphi(0) = 0
\]

\[
\kappa_1 e^2 \leq e\varphi(e) \leq \kappa_2 e^2, \forall e \neq 0
\]

**Theorem 3.1** If \( 0 < \alpha_i < 1 \) and \( 0 < \delta < \delta' \), then \( \phi_i(e) \in F[\kappa_1, \kappa_2] \) and \( \phi_i'(e) \in F[0, \kappa_2 - \kappa_1] \), where

\[
\begin{align*}
\kappa_1 &= \lambda_i, \kappa_2 = \delta^{\alpha_i-1}, \\
\kappa_1 &= \delta^{\alpha_i-1}, \kappa_2 = \delta^{\alpha_i-1}, \\
\kappa_1 &= \delta^{\alpha_i-1}, \kappa_2 = \lambda_i,
\end{align*}
\]

\[
(44)
\]

**Proof.** As \( \alpha < 1 \) and \( \delta > 0 \), \( \forall e \neq 0 \), we have

\[
\begin{align*}
\delta^{\alpha_i-1} &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, |e| \leq \delta' \\
\frac{\delta^{\alpha_i-1}}{\delta^{\alpha_i-1}} &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, \forall e \neq 0 \\
\delta^{\alpha_i-1} &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, \forall e \neq 0
\end{align*}
\]

then we obtain

\[
\begin{align*}
\lambda_i &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, \lambda_i \leq \delta^{\alpha_i-1}, \forall e \neq 0 \\
\delta^{\alpha_i-1} &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, \lambda_i \leq \delta^{\alpha_i-1}, \forall e \neq 0 \\
\delta^{\alpha_i-1} &\leq \frac{\phi_i(e)}{e} \leq \delta^{\alpha_i-1}, \lambda_i \leq \delta^{\alpha_i-1}, \forall e \neq 0
\end{align*}
\]

then we have

\[
\begin{align*}
\lambda_i e^2 &\leq \phi_i(e) \leq \delta^{\alpha_i-1} e^2, \lambda_i \leq \delta^{\alpha_i-1} \\
\delta^{\alpha_i-1} e^2 &\leq \phi_i(e) \leq \delta^{\alpha_i-1} e^2, \lambda_i \leq \delta^{\alpha_i-1} \\
\delta^{\alpha_i-1} e^2 &\leq \phi_i(e) \leq \delta^{\alpha_i-1} e^2, \lambda_i \leq \delta^{\alpha_i-1}
\end{align*}
\]

As \( \phi_i(0) = 0 \), the two conditions of the Definition 3.1 are satisfied. Therefore, \( \phi_i(e) \in F[\kappa_1, \kappa_2] \), where the \( \kappa_1, \kappa_2 \) are given in Theorem 3.1. Obviously, \( \phi_i'(e) \in F[0, \kappa_2 - \kappa_1] \).

Q.E.D.

On the basis of (41) and (44), absolute stability can be preferentially performed. In another paper, we have proposed LMI-based methods to analyze the absolute stability of system like (41), thus, the process is omitted here.

**IV. APPLICATION TO THE BALL-BEAM SYSTEM**

This section introduces the ball-beam system and its mathematical model, on the basis of which LADRC, NLADRC and SADRC are designed. Then a series of hardware experiments are implemented to verify the performance of the above controllers.

**A. The Ball-Beam System and Its Mathematical Model**

The experiments are carried out on the ball-beam system of GBB1004 from Googol Technology Ltd, shown in Fig. 3. The position of the ball is measured by the voltage of the potentiometer, of which the range is 0~5 V and its corresponding position is 0~400 cm. The motor is DC motor. The motion control board is of the type IPM100, reading signals from the potentiometer and the encoder, communicating with the computer and supplying control voltage to the power amplifiers. Because the IPM100 supports real-time operations, the control strategy is implemented on the Matlab Simulink platform. The sampling time and the solver were set to be 0.02s and ode1(Euler), respectively.

![Fig. 3. Experimental setup of the ball-beam system.](image)

It should be pointed out that there is an imbedded PID controller in this product to control the position of the DC servo motor with quickness and no overshoot. Therefore, we only consider the dynamic model of ball, which is given as follows [27]:

\[
\left( \frac{J}{L^2} + m \right) \ddot{\gamma} + mg \sin \alpha - m \dot{\gamma} \dot{\alpha}^2 = 0
\]

where \( \alpha \) is the beam’s incline angle, \( g \) is acceleration of gravity, \( m \) is the ball’s mass, \( J \) is the ball’s moment of inertia, \( \gamma \) is the ball’s position along the beam, \( R \) is the ball’s radius.

Assume that the movement of the ball is roll, and we neglect the friction. \( \theta \) can be treated as the control input \( u \). Since the expected angle \( \alpha \) is 0, we can linearize the Eq. (48) near 0 angle and we obtain the following equation [27]:

\[
\ddot{\gamma} = - \frac{mg}{(J / L^2 + m)} \alpha = - \frac{mgd}{L' (J / L^2 + m)} \dot{\theta} = - \frac{mgd}{L' (J / L^2 + m)} u
\]

The parameters in (49) are \( m=0.11 \text{kg}, R=0.015 \text{m}, g=9.8 \text{m/s}^2, L'=0.4 \text{m}, d=0.04 \text{m}, J = 2mR^2/5 \).

**B. ADRC Design**

Based on the nominal second order dynamic model (49), we design an ADRC controller. Firstly, make use of the following second order tracking differentiator (TD) to generate TP

\[
\begin{align*}
\dot{v}_1 &= v_2 \\
\dot{v}_2 &= \lambda^2 (r - v_1) - 2\lambda v_2
\end{align*}
\]

where \( \lambda \) can be treated as an adjustable speed factor, and the bigger the \( \lambda \), the faster the \( v_1 \) convergence to \( r \). Moreover, there is no overshoot in the process of convergence. Note that
the initial value of $v_1$ should be set as $v_1 = 0.4$, because the initial position of the ball is always fixed in 0.4.

Here we design a third order ESO as follows

$$\begin{cases}
    e = z_1 - y \\
    \dot{z}_1 = z_2 - \beta_1 \cdot \text{fal}(e, \alpha_1, \delta) \\
    \dot{z}_2 = z_3 - \beta_2 \cdot \text{fal}(e, \alpha_2, \delta) + b_0 \cdot u \\
    \dot{z}_3 = -\beta_3 \cdot \text{fal}(e, \alpha_3, \delta)
\end{cases} \tag{51}$$

where $b_0 = -\frac{mgd}{L'(\frac{1}{2} + r)} = \frac{5}{\tau}$, $z_i (i = 1, 2, 3)$ provide an estimation of $\gamma_\gamma$, and the total disturbance, respectively. In general, to avoid “peaking phenomenon”, we should make the initial error “$e$” be zero as much as possible. Therefore, in this experiment, the initial value of $z_1$ is generally set to be 0.4 m, and we will conduct an experiment to verify the “peaking phenomenon” [28]–[31] when there exists initial error.

The control law is designed as follows

$$u = \frac{u_0 - z_3}{b_0}, u_0 = \sum_{i=1}^{2} k_i (v_i - z_i) \tag{52}$$

The parameter values of ADRC are determined and listed as follows. TP: $\lambda = 0.5$; CL: $\omega_c = 1.5$, $k_1 = \omega_c^2$, $k_2 = 2 \omega_c$; LESO: $\omega_0 = 6$, $\beta_{01} = 3 \omega_0$, $\beta_{02} = 3 \omega_0^2$, $\beta_{03} = \omega_0^3$; NLSEO: $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.25$, $\delta = 0.002$, $\omega'_0 = 3$, $\beta'_{01} = 3 \omega'_0$, $\beta'_{02} = 3 \omega'_0 / 5$, $\beta'_{03} = \omega'_0^3 / 9$; SADRC: $\delta' = 0.006$.

### C. Hardware Experiments

A series of Hardware tests are carried out to check and compare the performance of LADRC, NLADRC and SADRC. We have considered step response with different initial conditions and adding step disturbance with different amplitude and the results are shown in Figs. 4 ~ 9, respectively. Note that in the figures there are many “jumps” in vertical coordinates, which is sensor noise and the real ball position is always smooth and slowly changed.

1) Step response: This step response experiment mainly investigates the influence of initial state errors to the performance of NLADRC (NLESO) and LADRC (LESO). Here we only compare LADRC with NLADRC, because that NLADRC is working in initial operation phase in the scheme of SADRC. Set the initial target position $\gamma = 0.2$m, then the target position changes from 0.2m to 0.3m at $t = 20$sec. The results of step response are shown in Fig. 4, which we can see that the results of LADRC and NLADRC are nearly same, except that the NLADRC is a little smoother.

We change the initial conditions of both LESO and NLESO and let $z_1 = 0$. Then we do the experiments again, and the results are shown in Fig. 5. We can see that the result of NLADRC is nearly the same with Fig. 4, but the result of LADRC turns clear deterioration. These two experiments show that the constant gain of LESO may cause “peaking phenomenon” and the “big error, small gain” of the NLESO is a very effective way to deal with the case that the initial conditions are not zeros. Note that there is friction between the ball and the beam, which is complex nonlinear and uncertain natural phenomenon and leads to plateau-like effect. Besides, due to the friction, noise and other unknown factors in real situations, the experimental results are a little different every time.

2) Add disturbance in the form of step signal with different amplitude: To compare the capability of anti-disturbance, we add a step signal as disturbance in the control input of the ball-beam system on the Matlab Simulink platform with amplitude of 0.6 rad at $t = 18$ sec. The results are shown in Fig. 6, from which we can see the performance of NLADRC and SADRC are nearly same and both superior to LADRC. Fig. 7 shows the corresponding control signals, which also show the consistent energy-efficiency advantages.

However, NLADRC is not always superior to LADRC. When the amplitude of the step signal turns big, the advantage of NLADRC turns small and even inferior to LADRC. We add a step signal as disturbance in the control input of the ball-beam system on the Matlab Simulink platform with amplitude of 1.5 rad and frequency of 1 second at $t = 18$ s. Fig. 8 shows that the control performance of SADRC is the best and LADRC is suboptimal, and NLADRC is the worst. Fig. 9 shows the control signals, which also show the consistent energy-efficiency advantages.

In summary, it can be concluded from the above two cases that NLADRC is suitable to work in circumstance with relative small disturbance while LADRC is suitable to work in circumstance with relative big disturbance. SADRC may be superior to both NLADRC and LADRC in some occasions and it can effectively deal with complex circumstances, which combines the advantages of both LADRC and NLADRC.

Overall, the above two groups of experiments verify the characteristics of LADRC, NLADRC and SADRC, which is
consistent with theory analysis. NLADRC shows advantages in occasions where the initial state errors are not zeros and the total disturbance is relatively small. LADRC has advantages when the total disturbance is relative big. SADRC has significant advantages when the total disturbance is complex, which combines the merits of both LADRC and NLADRC.

V. CONCLUSION

In this paper, simple parameter tuning principles and formulas have been proposed for the NLESOS, which make it much easier for practical application. It is shown that the NLADRC is not always superior to LADRC and that the performance is limited by the boundary as a function of the amplitude and derivative of the total disturbance. After illustrating the characteristics of both LADRC and NLADRC, a SADRC scheme has been proposed, which combines the advantages of LADRC and NLADRC. Stability characteristics of the proposed design is also analyzed. The ball-beam system is used as a platform to verify the SADRC scheme.

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