ACTIVE DISTURBANCE REJECTION CONTROL FOR REJECTING BOUNDARY DISTURBANCE FROM MULTI-DIMENSIONAL KIRCHHOFF PLATE VIA BOUNDARY CONTROL

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Abstract. In this paper, an algorithm is developed to reject time and spatially varying boundary disturbances from a multi-dimensional Kirchhoff plate via boundary control. The disturbance and control input are assumed to be matched. The active disturbance rejection control (ADRC) approach is adopted for developing the algorithm. A state feedback scheme is designed to estimate the disturbance based on an infinite number of ordinary differential equations obtained from the original multi-dimensional system using infinitely many time-dependent test functions. The proposed control law cancels the disturbance using its estimated value. All subsystems in the closed-loop are shown to be asymptotically stable. Simulation results are presented to validate the theoretical conclusions and to exhibit the reduction in the peaking phenomenon due to the use of time varying gains instead of constant high gains.

Key words. Kirchhoff plate, boundary control, disturbance rejection, stabilization.

AMS subject classifications. 74K20, 93C20, 93C25

1. Introduction and problem formulation. Roughly speaking, the real value of modern control theory lies in its capacity of rejecting disturbances. In this regard, the design of state feedback controllers plays an important role in control theory. In the past three decades, many different approaches have been developed to deal with uncertainties such as internal model principle for output regulation, robust control for systems with uncertainties from un-modeled dynamics and external disturbances, adaptive control for systems with unknown parameters, to name just a few. Most of these approaches however, focus on the worst case scenario which makes the controller designed conservative. The active disturbance rejection control (ADRC), as an unconventional design strategy similar to the external model principle ([25]), was first proposed by Han in 1998 based on realistic rethinking about the PID technology that has dominated the control engineering for almost one century ([17, 30]). The uncertainties dealt with by ADRC are much more complicated. For instance, ADRC can deal with the coupling between the external disturbances, the system un-modeled dynamics, and the superadded unknown part of control input. The most remarkable feature of ADRC is that the disturbance is estimated, in real time, through an extended state observer ([9]) and is canceled in the feedback loop. This reduces the control energy significantly in practice ([35]). The convergence of ADRC for general nonlinear lumped parameter systems was established recently in [10].
tion of ADRC to the systems described by one-dimensional partial differential equations (PDEs) is in our recent works [11, 12, 13] but the ADRC for multi-dimensional PDEs has not yet been studied.

Many of the aforementioned control methods have also been used to deal with uncertainties in PDEs. Sliding mode control (SMC), which is an inherently robust technique, is a popular approach to disturbance rejection from infinite-dimensional systems. But often, it requires the input and output operators of the plant to be bounded and hence does not apply to the boundary control of PDEs ([26]). It is only very recently, that boundary SMC controllers have been designed for one-dimensional heat, wave, Euler-Bernoulli, Schrödinger equations with boundary input disturbances, see in [6, 11, 12, 13]. In [14, 21], adaptive controllers are designed for one-dimensional wave equations by treating the unknown constants describing the disturbance as uncertain parameters. For adaptive control of PDEs, we also refer to [20] where an adaptive design is introduced for the first time for handling one-dimensional parabolic PDEs with disturbance and anti-damping. For a result on the robust control of PDEs, see [3]. For a stochastic PDE, an optimal control problem constrained by uncertainties in system and control is addressed in [29]. Another powerful method in dealing with uncertainties is based on the Lyapunov functional approach. In [8], a boundary control is designed by the Lyapunov method for one-dimensional Euler-Bernoulli beam equation with spatial and boundary disturbance. The internal model principle is also generalized to infinite-dimensional systems [19, 28, 36]. In [28], the tracking and disturbance rejection problem for infinite-dimensional linear systems, with reference and disturbance signals that are finite superpositions of sinusoids, are considered. The results are applied to some PDEs including the noise reduction in a structural acoustics model described by a two-dimensional PDE. An interesting PDE example in [28] is on the disturbance rejection in a coupled beam where the disturbance and control are not matched. However, there are not so many works, to the best of our knowledge, for the stabilization (instead of reference tracking) of multi-dimensional PDEs with disturbance. In our recent work [15], the stabilization of a multi-dimensional wave equation with disturbance is addressed.

In this paper, we are concerned with boundary stabilization of the following multi-dimensional Kirchhoff equation with Neumann boundary control and control matched external disturbance:

$$\begin{cases}
w_{tt}(x, t) - \gamma \Delta w_t(x, t) + \Delta^2 w(x, t) = 0, x \in \Omega, t > 0, \\
w(x, t)|_{\Gamma} = 0, t \geq 0, \\
\Delta w(x, t)|_{\Gamma} = u(x, t) + d(x, t), t \geq 0, \\
w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), x \in \Omega,
\end{cases}$$

where \( \Omega \subset \mathbb{R}^n \) is an open bounded domain with smooth boundary of \( C^4 \)-class ([23]), \( \gamma > 0 \) is usually a small number by which the Euler-Bernoulli plate is the limit case of the Kirchhoff plate as \( \gamma \to 0 \), \( u \) is the control input, \( (w_0, w_1) \) is the initial state, and \( d \) is the unknown external disturbance which is supposed to satisfy that

$$d \in L^\infty(0, \infty; C(\Gamma)) \cap C(0, \infty; C(\Gamma)), d_t \in L^\infty(0, \infty; C(\Gamma)).$$

The Kirchhoff plate is originally a two-dimensional mathematical model that is used to determine the stresses and deformations in thin plates subjected to forces and moments. This theory is an extension of the Euler-Bernoulli plate beam theory. The
theory assumes that a mid-surface plane can be used to represent a three-dimensional plate in two-dimensional form. In one-dimensional case, it reduces to the Rayleigh beam by adding the rotary inertia effects to the Euler-Bernoulli beam ([16]).

We consider system (1.1) in the energy Hilbert state space \( \mathcal{H} = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \). Throughout the paper, we use \( w_t \) or \( \dot{w} \) to denote the time derivative of \( w \) with respect to \( t \). It is well known that when there is no disturbance, the collocated feedback control

\[
(1.3) \quad u(x, t) = -k \frac{\partial w_t(x, t)}{\partial \nu}, \quad x \in \Gamma, \ t \geq 0, \ k > 0
\]

exponentially stabilizes the system (1.1) provided that there exists a coercive smooth vector field \( h \) on \( \Omega \), that is, the following condition is satisfied (Theorem 1.5 of [23]):

\[
(1.4) \quad \begin{cases} \quad \text{(i). } h \text{ is parallel to } \nu \text{ on } \Gamma, \ h(\sigma) = k(\sigma)\nu(\sigma) \text{ for a smooth } k, \sigma \in \Gamma. \\ \quad \text{(ii). For some constant } \rho > 0 \text{ and all vectors } y \in (L^2(\Omega))^n : \int_{\Omega} H(x) y(x) \cdot y(x) dx \geq \rho \int_{\Omega} |y(x)|^2 dx \text{ where } H(x) = \{\partial h_i/\partial x_j\}_{i,j=1}^n. \end{cases}
\]

For \( n = 2 \) this geometrical condition can be removed ([18]). However, the stabilizing control law (1.3) is not robust to the external disturbance, which is seen from the following example.

**Example 1.1.** Let \( \Omega = \{x = (x_1, x_2) \in \mathbb{R}^2| \ x_1^2 + x_2^2 < 1\} \) be a two-dimensional disk and \( \partial \Omega = \Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2| \ x_1^2 + x_2^2 = 1\} \). Let \( d(x, t) \equiv d \) be a constant disturbance. It is directly verified that system (1.1) under feedback (1.3) admits the following solution:

\[
(1.5) \quad (w(x, t), w_t(x, t)) = \left(\frac{d(x_1^2 + x_2^2) \ln(x_1^2 + x_2^2)}{8}, 0\right),
\]

which shows that in the presence of disturbance, the control must be re-designed.

Now, we formulate the problem into a standard abstract second order system ([4]). Let \( A \) be the positive self-adjoint operator in \( L^2(\Omega) \) defined by

\[
(1.6) \quad A\phi = \Delta^2 \phi, \quad D(A) = \{\phi \in H^4(\Omega) | \phi|_\Gamma = \Delta \phi|_\Gamma = 0\}.
\]

By interpolation result in theorem 11.6, Chapter 1 of [24], we have the following space identifications:

\[
(1.7) \quad \begin{cases} \quad D(A^0) = H^4(\Omega) \cap H_0^1(\Omega), \quad \frac{1}{8} < \theta < \frac{5}{8}, \\ \quad D(A^\theta) = \{\phi \in H^4(\Omega) | \phi|_\Gamma = \Delta \phi|_\Gamma = 0\}, \quad \frac{5}{8} < \theta \leq 1. \end{cases}
\]

In particular (see also [23]),

\[
(1.8) \quad \begin{cases} \quad A^{1/2} \phi = -\Delta \phi, \quad D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega), \\ \quad D(A^{1/4}) = H_0^1(\Omega). \end{cases}
\]

We endow \( D(A^{1/4}) = H_0^1(\Omega) \) with the following equivalent inner product induced norm:

\[
(1.9) \quad \|\psi\|_{D(A^{1/4})}^2 = \|\psi\|_{L^2(\Omega)}^2 + \gamma \|
abla \psi\|_{L^2(\Omega)}^2 = \|1 + \gamma A^{1/2}\|^2 \psi\|_{L^2(\Omega)}^2, \forall \psi \in H_0^1(\Omega).
\]
In this way, the state space is just \( \mathcal{H} = D(A^{1/2}) \times D(A^{1/4}) \) and the inner product induced norm in \( \mathcal{H} \) is given by
\[
\| (f, g) \|_2^2 = \| \Delta f \|_{L^2(\Omega)}^2 + \| (1 + \gamma A^{1/2})^{1/2} g \|_{L^2(\Omega)}^2
\]
\[
= \| \Delta f \|_{L^2(\Omega)}^2 + \| g \|_{L^2(\Omega)}^2 + \gamma \| \nabla g \|_{L^2(\Omega)}^2, \forall (f, g) \in \mathcal{H}.
\]
The control space is taken as usual as \( U = L^2(\Gamma) \). Define the operator \( \mathcal{A} : D(\mathcal{A}) \to \mathcal{H} \) as follows:
\[
(1.10) \quad \mathcal{A}(f, g) = (g, -(1 + \gamma A^{1/2})^{-1} A f), \forall (f, g) \in D(\mathcal{A}),
\]
\[
D(\mathcal{A}) = \{ (f, g) \in \mathcal{H} \cap (H^4(\Omega) \times H^2(\Omega)) \mid \Delta f|_\Gamma = 0, g|_\Gamma = 0 \}.
\]
It is easy to verify that \( \mathcal{A}^* = -\mathcal{A} \) in \( \mathcal{H} \). Set
\[
(1.11) \quad \mathcal{A} = (1 + \gamma A^{1/2})^{-1} A, \quad D(\mathcal{A}) = D(A).
\]
It is obvious that \( \mathcal{A} \) is a positive definite unbounded operator in \( L^2(\Omega) \). The following Gelfand’s triple inclusions are valid:
\[
(1.12) \quad D(A^{1/2}) \hookrightarrow D(A^{1/4}) \hookrightarrow D(A^{1/2})',
\]
where \( D(A^{1/2})' \) is the dual space of \( D(A^{1/2}) \) considering \( D(A^{1/4}) \) as the pivot space. An extension \( \mathcal{A} \in \mathcal{L}(D(A^{1/2}), D(A^{1/2})') \) of \( \mathcal{A} \) is defined by
\[
(\mathcal{A} f, g)_{D(A^{1/2})' \times D(A^{1/2})} = ((1 + \gamma A^{1/2})^{-1/2} A^{1/2} f, (1 + \gamma A^{1/2})^{-1} A^{1/2} g)_{D(A^{1/4})},
\]
\[\forall f, g \in D(A^{1/2}).\]
Define the Green map \( \Upsilon \in \mathcal{L}(L^2(\Gamma), H^{5/2}(\Omega)) \) ([24, p.188-189]), i.e., \( \Upsilon(u + d) = v \) if and only if
\[
(1.13) \quad \begin{cases} 
\Delta^2 v(x) = 0, & x \in \Omega, \\
|v|_\Gamma = 0, & \Delta v|_\Gamma = u + d.
\end{cases}
\]
Using the map \( \Upsilon \), one can write (1.1) in \( D(A^{1/2})' \) as
\[
(1.14) \quad \tilde{w} + \mathcal{A}(w - \Upsilon(u + d)) = 0,
\]
which is further written as
\[
(1.15) \quad \tilde{w} = -\mathcal{A} w + B(u + d),
\]
where \( B \in \mathcal{L}(U, D(A^{1/2})') \) is given by
\[
(1.16) \quad B u_0 = \mathcal{A} \Upsilon u_0, \quad \forall u_0 \in U.
\]
Define \( B^* \in \mathcal{L}(D(A^{1/2}), U) \), the adjoint of \( B \), by
\[
(1.1) \quad \langle B^* f, u_0 \rangle_U = \langle f, B u_0 \rangle_{D(A^{1/2})' \times D(A^{1/2})'}, \quad \forall f \in D(A^{1/2}), u_0 \in U.
\]
Then for any \( f \in D(A) \) and \( u_0 \in C_0^\infty(\Gamma) \),
\[
\langle f, B u_0 \rangle_{D(A^{1/2})' \times D(A^{1/2})'} = \langle (1 + \gamma A^{1/2})^{-1} A f, \mathcal{A}^{-1} B u_0 \rangle_{D(A^{1/4})}
\]
\[
= \langle A f, \Upsilon u_0 \rangle_{L^2(\Omega)} = \langle A f, v_0 \rangle_{L^2(\Omega)} = \left\langle \frac{\partial f}{\partial \nu}, u_0 \right\rangle_U,
\]
where $v_0 = Y u_0$. Since $C_0^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, we finally obtain

\begin{equation}
(1.17) \quad B^* f = \frac{\partial f}{\partial \nu}|_\Gamma, \forall f \in D(A^{1/2}).
\end{equation}

Therefore, system (1.1) can be written as

\begin{equation}
(1.18) \quad \frac{d}{dt} \begin{pmatrix} w \\ w_t \end{pmatrix} = A \begin{pmatrix} w \\ w_t \end{pmatrix} + B(u + d),
\end{equation}

where $B = (0, B)^\top : U \to H$, $B^* : H \to U$ is determined by $\langle Bu_0, F \rangle_H = \langle u_0, B^* F \rangle_U$ for all $u_0 \in U$ and $F = (f, g)^\top \in D(A^{1/2}) \times D(A^{1/2})$, which is given by

\begin{equation}
(1.19) \quad B^* (f, g)^\top = \frac{\partial g}{\partial \nu}|_\Gamma, \forall (f, g)^\top \in \left[ H^2(\Omega) \cap H^1_0(\Omega) \right]^2.
\end{equation}

Since $B$ is admissible to the $C_0$-semigroup generated by $A$ (see e.g., [33] and [22, p.993]), the solution of (1.18) is understood in the sense of

\begin{equation}
(1.20) \quad \frac{d}{dt} \langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle_H = \langle \begin{pmatrix} w \\ w_t \end{pmatrix}, A^* \begin{pmatrix} f \\ g \end{pmatrix} \rangle_H
+ \int_{\Gamma} [u(x, t) + d(x, t)] \frac{\partial g(x)}{\partial \nu} \, dx, \forall (f, g)^\top \in D(A^*).
\end{equation}

The main contributions of this paper are: a) introduce a state feedback to estimate the general control matched disturbance that depends on both time and spatial variable by ADRC approach to multi-dimensional PDEs; b) introduce a time varying gain to achieve the disturbance rejection and the peaking value reduction.

We proceed as follows. In Section 2, we deal with a simple case of $d(x, t) = d(t)$. We use this case as a heuristic example to explain the whole process of ADRC in dealing with disturbance. Some preliminary mathematical formulations are also presented in this section. The main results for general disturbance are stated in Section 3. Section 4 is devoted to the proof of the main results. In Section 5, we present some numerical simulations for illustration.

2. Preliminary: Case study of $d(x, t) = d(t)$. To facilitate a rudimentary understanding of ADRC, we consider the case $d(x, t) = d(t)$, i.e. the disturbance is a function of time only, for the following reasons: a) the essence of ADRC is first shown with simplicity and clarity, while the general case discussed in the next section; b) the same mathematical reasoning can be clearly presented for this rather simple case before the tedious computations and estimations are shown in the more general case; c) this simple case is significant in reality by itself but it is obscurely hidden in the sophistication of the general case.

The first objective is to estimate the disturbance $d$ through a state feedback estimator (extended state observer in ODEs), a key part of the ADRC. To this purpose, let $g \in H^4(\Omega)$ be the solution of the following elliptic boundary-value problem ([24, p.188]):

\begin{equation}
(2.1) \quad \begin{cases}
\Delta^2 g(x) = 0, & x \in \Omega, \\
g|_\Gamma = 0, \frac{\partial g}{\partial \nu}|_\Gamma = 1.
\end{cases}
\end{equation}
Substitute \((f, g)^\top = (0, g)^\top \in D(A^*)\) into (1.20) to obtain
\[
\frac{d}{dt} \int_\Omega [w_t(x, t) g(x) + \gamma \nabla w_t(x, t) \cdot \nabla g(x)] dx
\]
\[
= - \int_\Omega \Delta w(x, t) \Delta g(x) dx + \int_\Gamma [u(x, t) + d(t)] \frac{\partial g(x)}{\partial \nu} dx
\]
\[
= - \int_\Omega \Delta w(x, t) \Delta g(x) dx + \int_\Gamma u(x, t) dx + \text{meas}(\Gamma) d(t),
\]
where \(\text{meas}(\Gamma)\) denotes the Lebesgue measure of \(\Gamma\) in space \(\mathbb{R}^{n-1}\). Set
\[
y(t) = \int_\Omega [w_t(x, t) g(x) + \gamma \nabla w_t(x, t) \cdot \nabla g(x)] dx, \quad y_0(t) = - \int_\Omega \Delta w(x, t) \Delta g(x) dx.
\]
Then (2.2) becomes
\[
\dot{y}(t) = y_0(t) + \int_\Gamma u(x, t) dx + \text{meas}(\Gamma) d(t).
\]
It is seen that (2.4) is just an ODE where the disturbance appears on the right side. This is the starting point to estimate the disturbance following the way for lumped parameter systems presented in [9]. It is realized by the following time varying high gain extended state observer for ODE system (2.4):
\[
\begin{align*}
\dot{\hat{y}}(t) &= y_0(t) + \int_\Gamma u(x, t) dx + \text{meas}(\Gamma) \hat{d}(t) - r(t)[\hat{y}(t) - y(t)], \\
\dot{\hat{d}}(t) &= - \frac{r^2(t)}{\text{meas}(\Gamma)} [\hat{y}(t) - y(t)],
\end{align*}
\]
where \(r \in C(\overline{\mathbb{R}}^+, \mathbb{R}^+)\) is a time varying gain that is required to satisfy
\[
\dot{r}(t) > 0, \quad \lim_{t \to \infty} r(t) = \infty, \quad \frac{\dot{r}(t)}{r(t)} \leq \overline{M}, \forall t \geq 0 \text{ for some } \overline{M} > 0.
\]
The following Lemma 2.1 is about the convergence of extended state observer (2.5) for system (2.4). In consequence, we can regard \(\hat{d}\) as an approximation of \(d\).

**Lemma 2.1.** Suppose that (1.2) and (2.6) hold. Let \(y_0\) and \(y\) be defined by (2.3) and \(g\) by (2.1). Then the solution of (2.5) satisfies
\[
\lim_{t \to \infty} |\hat{y}(t) - y(t)| = \lim_{t \to \infty} |\hat{d}(t) - d(t)| = 0.
\]
**Proof.** Let
\[
\tilde{y}(t) = r(t)[\hat{y}(t) - y(t)], \quad \tilde{d}(t) = \hat{d}(t) - d(t)
\]
be the errors. Then, it follows from (2.4) and (2.5) that \((\tilde{y}, \tilde{d})\) satisfies
\[
\begin{align*}
\dot{\tilde{y}}(t) &= -r(t)\tilde{y}(t) + \text{meas}(\Gamma) r(t) \tilde{d}(t) + \frac{\dot{r}(t)}{r(t)} \tilde{y}(t), \\
\dot{\tilde{d}}(t) &= - \frac{r(t)}{\text{meas}(\Gamma)} \tilde{y}(t) - \tilde{d}(t).
\end{align*}
\]
We construct a Lyapunov function for system (2.9):

$$\text{(2.10)} \quad V(y_1, y_2) = (y_1, y_2)P(y_1, y_2)^T,$$

where the $2 \times 2$ positive definite matrix $P$ is the solution of the following Lyapunov equation:

$$F^TP + PF = -I_{2 \times 2}, F = \left( \begin{array}{c} -\frac{1}{\text{meas}(\Gamma)} \\ 0 \end{array} \right).$$

Notice that

$$\text{(2.11)} \quad \lambda_{\min}(P)\|y_1, y_2\|^2 \leq V(y_1, y_2) \leq \lambda_{\max}(P)\|y_1, y_2\|^2, \forall (y_1, y_2) \in \mathbb{R}^2,$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the maximal and minimal eigenvalues of $P$, respectively. Finding the derivative of $V$ along the solution of (2.9) yields

$$\dot{V}(t) = \left( \ddot{y}(t), \dot{d}(t) \right) P \left( \dot{y}(t), \dot{d}(t) \right)^T + \left( \dot{y}(t), \dot{d}(t) \right) P \left( \ddot{y}(t), \dot{d}(t) \right)^T$$

$$\leq r(t) \left( \ddot{y}(t), \dot{d}(t) \right) \left[ PF + F^T P \right] \left( \dot{y}(t), \dot{d}(t) \right)^T$$

$$+ \left( \ddot{y}(t), \dot{d}(t) \right) P \left( \frac{r(t)}{r(t)} \ddot{y}(t), -\dot{d}(t) \right)^T + \left( \frac{r(t)}{r(t)} \ddot{y}(t), -\dot{d}(t) \right) P \left( \ddot{y}(t), \dot{d}(t) \right)^T$$

$$\leq -r(t) \left\| \left( \ddot{y}(t), \dot{d}(t) \right) \right\|_{\mathbb{R}^2}^2 + N_1 \left\| \left( \dot{y}(t), \dot{d}(t) \right) \right\|_{\mathbb{R}^2}^2 + N_2 \left\| \left( \ddot{y}(t), \dot{d}(t) \right) \right\|_{\mathbb{R}^2}^2,$$

where $N_1$ and $N_2$ are two positive constants and $V(t) = V(\dot{y}(t), \dot{d}(t))$ is the solution of (2.9). In the last step of (2.12), the boundedness of $\ddot{d}$ was used. This together with (2.11) gives

$$\text{(2.13)} \quad \frac{dV(t)}{dt} \leq -\frac{r(t)}{\lambda_{\max}(P)} V(t) + \frac{N_1}{\lambda_{\min}(P)} V(t) + \frac{N_2}{\sqrt{\lambda_{\min}(P)}} \sqrt{V(t)}.$$ 

Since $\lim_{t \to \infty} r(t) = +\infty$, there exists $t_0 > 0$ such that $r(t) > \frac{2\lambda_{\max}(P)}{\lambda_{\min}(P)} N_1$ for all $t \geq t_0$. This together with (2.13) shows that

$$\text{(2.14)} \quad \frac{d\sqrt{V(t)}}{dt} \leq -\frac{1}{4\lambda_{\max}(P)} r(t) \sqrt{V(t)} + \frac{N_2}{2\sqrt{\lambda_{\min}(P)}} \forall t \geq t_0,$$

and hence

$$0 \leq \lim_{t \to \infty} \sqrt{V(t)} \leq \lim_{t \to \infty} e^{-\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma) d\sigma} \sqrt{V(t_0)}$$

$$+ \frac{N_2}{2\sqrt{\lambda_{\min}(P)}} \int_{t_0}^{t} e^{-\int_{s}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma) d\sigma} ds$$

$$\leq \lim_{t \to \infty} 2N_2 \lambda_{\max}(P) \frac{\sqrt{V(t_0)}}{\sqrt{\lambda_{\min}(P)}} \frac{e^{\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma) d\sigma}}{r(t) e^{-\int_{t_0}^{t} \frac{1}{4\lambda_{\max}(P)} r(\sigma) d\sigma}} = 0.$$
In the last step of (2.15), the L'Hospital rule and assumption (2.6) were used. Therefore, \( \lim_{t \to \infty} V(t) = 0 \) which in turn deduces from (2.11) that

\[
\tilde{y}(t) \to 0, \quad \tilde{d}(t) \to 0 \text{ as } t \to \infty.
\]

This shows (2.7) by (2.8) and assumption (2.6).  \( \square \)

We point out that our disturbance estimation is for unstable system (1.1). The next objective of the ADRC is to cancel disturbance in the feedback-loop. Since the collocated feedback control \( u(x, t) = -k \frac{\partial w(t, x)}{\partial \nu} \) stabilizes system (1.1) without disturbance, a stabilizing control law to (1.1) is naturally designed as follows:

\[
u(x, t) = -k \frac{\partial w_t(x, t)}{\partial \nu} - \hat{\tilde{d}}(t), \quad k > 0.
\]

It is seen that the second term in the right side of (2.17) is used to cancel the effect of the disturbance. This is just the estimation/cancellation nature of ADRC. Under feedback (2.17), the closed-loop system of (1.1) becomes

\[
\begin{align*}
\begin{cases}
\dot{w}(x, t) - \gamma \Delta w_t(x, t) + \Delta^2 w(x, t) = 0, & x \in \Omega, t > 0, \\
w(x, t)|_{\Gamma} = 0, \\
\Delta w(x, t)|_{\Gamma} = -k \frac{\partial w_t(x, t)}{\partial \nu} - \tilde{d}(t) + d(t), \\
\dot{\tilde{y}}(t) = y_0(t) - \kappa \int_{\Gamma} \frac{\partial w_t(x, t)}{\partial \nu} \, dx - \gamma(t)[\tilde{y}(t) - y(t)], \\
\dot{\tilde{d}}(t) = -\frac{r^2(t)}{\text{meas}(\Gamma)}[\tilde{y}(t) - y(t)],
\end{cases}
\end{align*}
\]

where \( g \) is defined by (2.1) and \( r \) is by (2.6).

Now, we are in a position to show the convergence of closed-loop system (2.18).

THEOREM 2.2. Assume conditions (1.2) and (1.4). Let \( y_0 \) and \( y \) be defined by (2.3) and \( g \) by (2.1). Then, system (2.18) is asymptotically stable in the sense of

\[
\lim_{t \to \infty} E(t) = \lim_{t \to \infty} \left[ \int_{\Omega} [\Delta w(x, t)]^2 + |w_t(x, t)|^2 + \gamma |\nabla w_t(x, t)|^2 + |\tilde{y}(t)| + |\tilde{d}(t) - d(t)| \right] = 0.
\]

Proof. Using the error variable \((\tilde{y}, \tilde{d})\) defined in (2.8), we write the equivalent form of system (2.18) as

\[
\begin{align*}
\begin{cases}
\dot{w}(x, t) - \gamma \Delta w_t(x, t) + \Delta^2 w(x, t) = 0, & x \in \Omega, t > 0, \\
w(x, t)|_{\Gamma} = 0, \\
\Delta w(x, t)|_{\Gamma} = -k \frac{\partial w_t(x, t)}{\partial \nu} - \tilde{d}(t), \\
\dot{\tilde{y}}(t) = -r(t)\tilde{y}(t) + \text{meas}(\Gamma)r(t)\tilde{d}(t) + \frac{\dot{r}(t)}{r(t)}\tilde{y}(t), \\
\dot{\tilde{d}}(t) = -\frac{r(t)}{\text{meas}(\Gamma)}\tilde{y}(t) - \tilde{d}(t).
\end{cases}
\end{align*}
\]
The convergence of the “ODE part” in (2.19) has been proven in Lemma 2.1. We need only show the convergence of the “w part” of system (2.19), that is,

\begin{equation}
\lim_{t \to \infty} \int_{\Omega} \left[ ||\Delta w(x,t)||^{2} + |w_t(x,t)|^{2} + \gamma |\nabla w_t(x,t)|^{2} \right] dx = 0.
\end{equation}

This is because if (2.20) is true, then it follows from (2.16) that

\[ \bar{y}(t) = \frac{\bar{y}(t)}{r(t)} + y(t) = \frac{1}{r(t)} \bar{y}(t) + \int_{\Omega} [w_t(x,t)g(x) + \gamma \nabla w_t(x,t) \cdot \nabla g(x)] dx \to 0 \quad \text{as} \quad t \to \infty. \]

Now, we prove (2.20). Similar to (1.18), we can write the “w part” of (2.19) as

\begin{equation}
\begin{aligned}
\frac{d}{dt} \left( \begin{array}{c}
w \\
w_t
\end{array} \right) &= \mathbb{A} \left( \begin{array}{c}
w \\
w_t
\end{array} \right) + \mathbb{B} \tilde{d}_{t} \quad \text{in} \quad D(A^{1/2}) \times D(A^{1/2}),
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
\mathbb{A} \left( \begin{array}{c}
f \\
g
\end{array} \right) &= \left( -\mathcal{A} f + kBB^{*} g \right), \forall (f,g) \in D(\mathbb{A}), \\
D(\mathbb{A}) &= \{(f,g)^{T} \mid f,g \in D(A^{1/2}), \mathcal{A} f + kBB^{*} g \in D(A^{1/4})\}, \\
\mathbb{B} &= (0,-B)^{T}.
\end{aligned}
\end{equation}

We claim that (i). The operator $\mathbb{A}$ defined in (2.22) generates a $C_0$-semigroup of contractions on $\mathcal{H}$; (ii). $\mathbb{B}$ is admissible to the $C_0$-semigroup $e^{\mathcal{A}t}$ ([33]). We first show claim (i). Actually, for any $(f,g)^{T} \in D(\mathbb{A}),$

\begin{equation}
\begin{aligned}
\Re\langle \mathbb{A}(f,g)^{T},(f,g)^{T} \rangle_{\mathcal{H}} &= \Re\langle A^{1/2}g, A^{1/2}f \rangle_{L^{2}(\Omega)} \\
-\Re\langle \mathcal{A} f + kBB^{*} g, g \rangle_{D(A^{1/4})} \\
= \Re\langle (1 + \gamma A^{1/2})^{-\frac{1}{2}} A^{1/2} g, (1 + \gamma A^{1/2})^{-\frac{1}{2}} A^{1/2} f \rangle_{D(A^{1/4})} \\
-\Re\langle \mathcal{A} f + kBB^{*} g, g \rangle_{D(A^{1/4}),D(A^{1/2})} \\
= \Re\langle \mathcal{A} f, g \rangle_{D(A^{1/2}),D(A^{1/2})} - \Re\langle \mathcal{A} f + kBB^{*} g, g \rangle_{D(A^{1/2}),D(A^{1/2})} \\
= -k\Re\langle BB^{*} g, g \rangle_{D(A^{1/2}),D(A^{1/2})} - k\Re\langle B^{*} g, B^{*} g \rangle_{U} = -k\|B^{*} g\|_{U}^{2} \leq 0,
\end{aligned}
\end{equation}

where in the third step of (2.23), the relation (1.12) was used. This shows that $\mathbb{A}$ is dissipative. Next, we show that $\mathbb{A}^{-1} \in \mathcal{L}(\mathcal{H})$. Solve the equation:

\[ \mathbb{A} \left( \begin{array}{c}
f \\
g
\end{array} \right) = \left( -\mathcal{A} f + kBB^{*} g \right) = \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right), \forall \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) \in \mathcal{H}
\]

to obtain $g = \phi \in D(A^{1/2}), -\mathcal{A} f - kBB^{*} g = \psi$. The latter is equivalent to

\[ \mathcal{A} f = -kBB^{*} \phi - \psi \in D(A^{1/2}). \]

Since $\mathcal{A}$ is isometric and surjective from $D(A^{1/2})$ to $D(A^{1/2})$, we find that

\[ f = \mathcal{A}^{-1}(-kBB^{*} \phi - \psi) \in D(A^{1/2}). \]

Hence

\begin{equation}
\mathbb{A}^{-1} \left( \begin{array}{c}
\phi \\
\psi
\end{array} \right) = \left( \mathcal{A}^{-1}(-kBB^{*} \phi - \psi) \right).
\end{equation}
By the Lumer-Phillips theorem [27, theorem 1.3.6], \(A\) generates a \(C_0\)-semigroup of contractions on \(H\). To prove claim (ii), we consider the following system

\[
\frac{d}{dt} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = A \begin{pmatrix} p \\ \dot{p} \end{pmatrix}.
\]

(2.25)

Since \(A\) generates a \(C_0\)-semigroup on \(H\) (as shown earlier), it follows that for any \((p(t, \cdot), \dot{p}(t, \cdot))^T \in D(A)\), the solution to (2.25) satisfies \((p(t), \dot{p}(t))^T \in D(A)\) for all \(t \geq 0\). Take the inner product on both sides of (2.25) with \((p(t), \dot{p}(t))^T\) and take (2.23) into account to obtain

\[
\text{Re}(\dot{p}, \dot{p})_{D(A^{1/4})} + \text{Re}(p, \dot{p})_{D(A^{1/2})} = -k\|B^*\dot{p}\|_U^2.
\]

that is, \(F(t) = \frac{1}{2} \left[\|A^{1/2}p\|_{L^2(\Omega)}^2 + \|\dot{p}\|_{D(A^{1/4})}^2\right]\) satisfies \(F(t) = -k\|B^*\dot{p}\|_U^2\). Therefore,

\[
k \int_0^T \|B^*\dot{p}\|_U^2 dt = F(0) - F(T) \leq F(0).
\]

This shows that \(B\) is admissible to the \(C_0\)-semigroup \(e^{At}\) ([34]). In other words, system (2.21) admits a unique solution \((w, \dot{w})^T \in H\) for any initial value \((w(\cdot, 0), \dot{w}(\cdot, 0))^T \in H\). Owing to (2.16), for any given \(\sigma > 0\), we may suppose that \(|\bar{d}(t)| \leq \sigma\) for all \(t > t_0\) for some \(t_0 > 0\). Now, we write the solution of (2.21) as

\[
\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = e^{At} \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} + \int_0^t e^{A(t-s)} \mathbb{B}\bar{d}(s)ds
\]

(2.26)

\[
= e^{At} \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} + e^{A(t-t_0)} \int_{t_0}^t e^{A(t-s-t_0)} \mathbb{B}\bar{d}(s)ds + \int_{t_0}^t e^{A(t-s-t_0)} \mathbb{B}\bar{d}(s)ds.
\]

The admissibility of \(\mathbb{B}\) implies that

\[
\left\| \int_0^t e^{A(t-s-t_0)} \mathbb{B}\bar{d}(s)ds \right\|_{L^2(\Omega)}^2 \leq C_1 \|\bar{d}\|_{L^2(\Omega, \Gamma)}^2 \leq C_1 t^2 \|\bar{d}\|_{L^\infty(0, \infty)}^2, \quad \forall t > 0
\]

for some constant \(C_t\) that is independent of \(\bar{d}\). On the other hand, under the assumption of the theorem, it is known that \(e^{At}\) is exponentially stable ([23]). By Remark 2.6 of [33], we have

\[
\left\| \int_{t_0}^t e^{A(t-s-t_0)} \mathbb{B}\bar{d}(s)ds \right\|_{L^\infty(\Gamma)} \leq L \|\bar{d}\|_{L^\infty(t_0, \infty)} \leq L\sigma,
\]

where \(L\) is a constant that is independent of \(\bar{d}\), and

\[
(u \mathcal{C} v)(t) = \begin{cases} u(t), & 0 \leq t \leq \tau, \\
v(t), & t > \tau.\end{cases}
\]

Suppose that \(\|e^{At}\| \leq L_0 e^{-\omega t}\) for some \(L_0, \omega > 0\). Then we have

\[
\left\| \begin{pmatrix} w(t, \cdot) \\ w_t(t, \cdot) \end{pmatrix} \right\|_H \leq L_0 e^{-\omega t} \left\| \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} \right\|_H + L_0 C_1 t_0^2 e^{-\omega(t-t_0)} \|\bar{d}\|_{L^\infty(0, t_0, L^2(\Gamma))} + L\sigma.
\]

(2.27)
Passing to the limit as $t \to \infty$ for (2.27), we finally obtain

$$\lim_{t \to \infty} \| \begin{pmatrix} w(\cdot, t) \\ w_1(\cdot, t) \end{pmatrix} \|_H \leq L\sigma.$$  

This proves (2.20). The proof is complete. $\Box$

At the end of this section, we point out that the time varying gains take many other advantages than the constant gains. The first is that the proper choice of the time varying gain can reduce significantly the notorious peaking value problem in high gain control, which will be explained numerically in Section 5. Here we indicate another advantage of time varying gain. This is about the boundedness of the derivative of the disturbance $|\ddot{d}|$ which is a necessary condition via constant gains yet not for other methods like sliding mode control (see, e.g., [11, 12, 13]). The following Proposition 2.3 shows that $d_1$ is allowed to grow exponentially at any growth rate.

**Proposition 2.3.** Suppose that the time varying gain $r$ satisfies condition (2.6). If $d$ and $r$ satisfies further that

$$\lim_{t \to \infty} |\ddot{d}(t)| = 0,$$

then the conclusions of Theorem 2.2 remain valid.

**Proof.** We need only show that

$$\lim_{t \to \infty} V(t) = 0,$$

where $V(t) = V(\tilde{y}(t), \tilde{d}(t))$ defined by (2.10), because other arguments are exactly the same as that in the proof of Theorem 2.2. Finding the derivative of $V$ along the solution of (2.9) yields

$$\dot{V}(t) \leq -r(t) \left\| \begin{pmatrix} \tilde{y}(t), \tilde{d}(t) \end{pmatrix} \right\|_{\mathbb{R}^2}^2 + \tilde{N}_1 \left\| \begin{pmatrix} \tilde{y}(t), \tilde{d}(t) \end{pmatrix} \right\|_{\mathbb{R}^2}^2$$

$$+ \tilde{N}_2 |\ddot{d}(t)| \left\| \begin{pmatrix} \tilde{y}(t), \tilde{d}(t) \end{pmatrix} \right\|_{\mathbb{R}^2},$$

where $\tilde{N}_1$ and $\tilde{N}_2$ are two positive constants. By (2.6), one can choose $t_0 > 0$ such that $r(t) > \frac{2\lambda_{\max}(P)}{\lambda_{\min}(P)} \tilde{N}_1$ for all $t \geq t_0$. This, together with (2.11) and (2.31), gives

$$\frac{d\sqrt{V(t)}}{dt} \leq -\frac{1}{4\lambda_{\max}(P)} r(t) \sqrt{V(t)} + \frac{\tilde{N}_2}{2\sqrt{\lambda_{\min}(P)}} |\ddot{d}(t)|, \forall t \geq t_0,$$

which yields further that

$$\sqrt{V(t)} \leq e^{-\int_{t_0}^t \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} \sqrt{V(t_0)}$$

$$+ \frac{\tilde{N}_2}{2\sqrt{\lambda_{\min}(P)}} \int_{t_0}^t |\ddot{d}(s)| e^{\int_{t_0}^s \frac{1}{4\lambda_{\max}(P)} r(\sigma)d\sigma} ds, \forall t \geq t_0.$$  

Passing to the limit as $t \to \infty$ in (2.33) by using the L’Hospital rule and assumption (2.29) gives (2.30). The proof is complete. $\Box$
The claim (2.30) by (2.29) enables us to include the disturbance like \( d(t) = \sin(t^n), n \geq 1 \) or even \( d(t) = \sin(e^t) \) in Theorem 2.2. This makes ADRC possess the same capacity as other methods like sliding mode control in dealing with the bounded disturbance without assuming the boundedness of the derivative for disturbance \([11, 12, 13]\). However, since (2.6) limits the exponential growth rate of \( r \), condition (2.29) limits that \( |\dot{d}| \) can grow at most exponentially although the exponential growth rate could be arbitrary by an appropriate choice of \( r \).

3. The main results for general \( d(x, t) \). In this section, we deal with the general case of disturbance \( d \) which depends on both time and space variables. To this purpose, we suppose additionally that \( d(\cdot, t) \) is H"older continuous with respect to \( x \) with the index \( \alpha \in (0, 1] \), that is, there exists a positive nondecreasing differentiable function \( K \) such that

\[
|d(x_1, t) - d(x_2, t)| \leq K(t)|x_1 - x_2|^\alpha, \ \forall \ x_1, x_2 \in \Gamma, \ t \geq 0.
\]

The examples of such kinds of disturbances include all finite sum of harmonic disturbances like \( d(x, t) = \sin(\alpha x) \) with \( K(t) = t + 1, \alpha = 1 \) and \( d(x, t) = \sin x \sin t \) with \( K = \alpha = 1 \). Actually, any periodic disturbance that is continuously differentiable with respect to \( x \) satisfies (3.1) with \( \alpha = 1 \). Let \( \varepsilon \) be a continuously differentiable function such that

\[
\varepsilon(t) \in (0, 1], \ \dot{\varepsilon}(t) < 0, \ \lim_{t \to \infty} \varepsilon(t) = 0.
\]

In addition, we can choose the appropriate \( \varepsilon \) and \( r \) satisfying (2.6) so that

\[
\delta(t) = \left(\frac{\varepsilon(t)}{K(t)}\right)^{1/\alpha} \text{ satisfies } \lim_{t \to \infty} r(t)\delta^{n+1}(t) = \infty \text{ and } \sup_{t > 0} |\dot{\delta}(t)\delta^n(t)| < \infty.
\]

To deal with the disturbance that depends on the spatial variable, we need the time varying covers of \( \Gamma \) so that the sum of measures for these covers is finite.

The following Lemma 3.1 was proved very recently in [15] (the proof is attached in Appendix).

**Lemma 3.1.** Let \( \delta \) be defined by (3.3). Then, one can find constructively a set of discrete points \( \{x^{(i)}\}_{i=1}^\infty \subset \Gamma \) so that the finite time varying covers \( \{\Gamma \cap U(x^{(i)}, \delta(t))\}_{i=1}^{N(t)} \) of \( \Gamma \) satisfy

\[
\Gamma \subset \{\Gamma \cap U(x^{(i)}, \delta(t))\}_{i=1}^{N(t)},
\]

\[
\sum_{i=1}^{N(t)} \text{meas}(\Gamma \cap U(x^{(i)}, \delta(t))) \leq C^n(\Gamma)(n-1)^{\frac{n+1}{2}} 2^{n-1} \text{meas}(\Gamma),
\]

where \( U(x^{(i)}, \delta(t)) \) denotes the ball in \( \mathbb{R}^n \) centered at \( x^{(i)} \in \Gamma \) with radius \( \delta(t) \), \( \text{meas}(\Gamma) \) is the Lebesgue measure of \( \Gamma \) in \( \mathbb{R}^{n-1} \), \( C(\Gamma) \) is a positive constant depending on \( \Gamma \) alone, and the time dependent integer \( N \) depends on \( \delta \) directly and \( \lim_{t \to \infty} N(t) = +\infty \).

The next step is to construct a state feedback estimator to estimate the disturbance by using the time varying covers in Lemma 3.1. But as opposed to (2.2), due to the spatial dependence of the disturbance, we will need an infinite number of equations...
as explained next. Let $(f_i^t, g_i^t) = (0, g_i^t) \in D(A^*)$, $i = 1, 2, \ldots$, so that

$$
\begin{align*}
\begin{cases}
  g_i^t|_\Gamma = 0, & \frac{\partial g_i^t(x)}{\partial \nu}|_{\Gamma U(x^{(i)}, \delta(t))} = 0, \\
  0 \leq \frac{\partial g_i^t(x)}{\partial \nu}|_{\Gamma U(x^{(i)}, \delta(t))} \leq \delta^2(t), \\
  |\partial g_i^t(x)|_{\Gamma U(x^{(i)}, \delta(t))} \leq C|\delta(t)|\delta(t),
\end{cases}
\end{align*}
$$

(3.5)

where $C$ is a fixed positive constant defined in (3.9) later. Such a function $g_i^t$ can be constructed by letting $g_i^t$ be the solution of the following elliptic boundary-value problem ([24, p.188]):

$$
\begin{align*}
\begin{cases}
  \Delta^2 g_i^t(x) = 0, & x \in \Omega, \\
  g_i^t|_\Gamma = 0, & \frac{\partial g_i^t(x)}{\partial \nu}|_\Gamma = \varphi_i^t|_\Gamma,
\end{cases}
\end{align*}
$$

(3.6)

where

$$
\varphi_i^t(x) = \frac{\delta^2(t)}{(\delta(1)^t)^n} \int_{\mathbb{R}^n} \chi_i(y) \exp \left( \frac{x - y}{\delta(1)^t} \right) dy, \quad \forall x \in \Gamma,
$$

(3.7)

$\chi_i$ is the characteristic function of $U(x^{(i)}, \frac{\delta(t)}{4})$ and $\varrho \in C^\infty(\Omega)$ is an mollifier given by ([1, p.36])

$$
\varrho(x) = \begin{cases}
  C_\varrho \exp\left(-1/(1 - |x|^2)\right), & \|x\|_{\mathbb{R}^n} < 1, \\
  0, & \|x\|_{\mathbb{R}^n} \geq 1,
\end{cases}
$$

(3.8)

where $C_\varrho$ is a normalization constant such that $\int_{\mathbb{R}^n} \varrho(x) dx = 1$. It is obvious that $\varphi_i^t$ is continuously differentiable with respect to $t$ and $\nabla \Delta \varphi_i^t$ is continuous with respect to $x$: $g_i^t$ is continuously differentiable with respect to $t$ and $C^\infty$-smooth respect to $x$ since it is a solution of elliptic problem (3.6) with $C^\infty$-smooth boundary value in $x$. Moreover, a simple computation shows that there exists a constant $C > 0$ such that

$$
|\varphi_i^t(x)| \leq C|\delta(t)|\delta(t), \quad \forall x \in \Gamma, \quad \|\nabla \Delta \varphi_i^t\|_{L^2(\Gamma)} \leq C\delta^{n-2}(t), \forall t \geq 0.
$$

(3.9)

Since $\frac{\partial g_i^t}{\partial \nu}|_\Gamma = \varphi_i^t|_\Gamma$ with $\varphi_i^t$ given by (3.7) is of the analytic expression, when we need $\frac{\partial g_i^t}{\partial \nu}|_\Gamma$, in what follows, we always write $\varphi_i^t|_\Gamma$ instead of $\frac{\partial g_i^t}{\partial \nu}|_\Gamma$. In addition, we emphasize that these constructed $g_i^t$ play the role of $g$ in (2.1) but since these functions are time variant, we need additional property (3.5) for $g_i^t$. Substitute $(f_i^t, g_i^t)^T = (0, g_i^t)^T \in D(A^*)$ into (1.20) to obtain

$$
\begin{align*}
\frac{d}{dt} \int_{\Omega} [w_t(x, t)g_i^t(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^t(x)] dx \\
= -\int_{\Omega} \Delta w(x, t)\Delta g_i^t(x) dx + \int_{\Omega} [u(x, t) + d(x, t)] \varphi_i^t(x) dx \\
+ \int_{\Omega} [w_t(x, t)g_i^t(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^t(x)] dx
\end{align*}
$$

(3.10)
where $\xi : [0, \infty) \to \Gamma \cap U(x^{(i)}, \delta(t))$ satisfies
\begin{align}
\left\{
\begin{array}{l}
d(\xi_i(t), t) = \frac{\int_\Gamma d(x, t)\varphi_i'(x)dx}{\int_\Gamma \varphi_i'(x)dx}, \\
\frac{d}{dt}(\int_\Gamma d(\xi_i(t), t)\int_\Gamma \varphi_i'(x)dx) = \int_\Gamma d_t(x, t)\varphi_i'(x)dx + \int_\Gamma d(x, t)\varphi_i'(x)dx, \\
\left| \int_\Gamma d(x, t)\varphi_i'(x)dx \right| \leq C'(\Gamma)\|d\|_{L^\infty(0, \infty; C(\Gamma))}\hat{\delta}(t)\delta^n(t)
\end{array}
\right.
\end{align}
for some constant $C'(\Gamma) > 0$. From these definitions, we can find an $M > 0$ such that
\begin{align}
\left\{
\begin{array}{l}
|d(\xi_i(t), t)| \leq M, \\
\left| \frac{d}{dt}(\int_\Gamma d(\xi_i(t), t)\int_\Gamma \varphi_i'(x)dx) \right| \leq M \\
+C'(\Gamma)\|d\|_{L^\infty(0, \infty; C(\Gamma))}\sup_{t>0}(\hat{\delta}(t)\delta^n(t)) < \infty.
\end{array}
\right.
\end{align}
Set, for $i = 1, 2, \ldots$, that
\begin{align}
\left\{
\begin{array}{l}
y_1(t) = \int_\Omega \left[ w_t(x, t)g_i^1(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^1(x) \right] dx, \\
y_2(t) = -\int_\Omega \Delta w(x, t)\Delta g_i^1(x)dx + \int_\Omega \left[ w_t(x, t)g_i^1(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^1(x) \right] dx.
\end{array}
\right.
\end{align}
Then, we have
\begin{align}
\dot{y}_i(t) = y_2(t) + \int_\Gamma u(x, t)\varphi_i'(x)dx + d(\xi_i(t), t)\int_\Gamma \varphi_i'(x)dx, i = 1, 2, \ldots.
\end{align}
It is seen that $g_i^1$ is contained in $y_1$ and $y_2$, defined by (3.13) but not explicitly in (3.14). The system (3.14), as an infinite number of ordinary differential equations, is our starting point to estimate the general disturbance $d$ motivated from the ADRC to lumped parameter systems ([9]). To this purpose, we design a time varying high gain extended state observer for system (3.14) as follows:
\begin{align}
\left\{
\begin{array}{l}
\dot{\hat{y}}_i(t) = y_2(t) + \int_\Gamma u(x, t)\varphi_i'(x)dx + \hat{d}_i(t)\int_\Gamma \varphi_i'(x)dx - r(t)[\hat{y}_i(t) - y_i(t)], \\
\frac{d}{dt}(\hat{d}_i(t)\int_\Gamma \varphi_i'(x)dx) = -r^2(t)[\hat{y}_i(t) - y_i(t)], i = 1, 2, \ldots,
\end{array}
\right.
\end{align}
where $r$ is defined in (2.6). We regard $\hat{d}_i(t)$ as an approximation of $d(\xi_i(t), t)$ which is confirmed by the following Lemma 3.2.

**Lemma 3.2.** Let $\{x^{(i)}\}$ be as defined in Lemma 3.1. Let $g_i^1$, $\hat{d}_i$, $d(\xi_i(t), t)$, and $(y_i, y_2)$ as introduced in the equations (3.5), (3.15), (3.11), and (3.13), respectively. Then
\begin{align}
\lim_{t \to \infty} |\hat{d}_i(t) - d(\xi_i(t), t)| = 0 \text{ uniformly for all } i = 1, 2, \ldots.
\end{align}
Now, the collocated like state feedback control law (1.1) is designed as:

$$u(x, t) = -k \frac{\partial w_t(x, t)}{\partial \nu} - \hat{d}(x, t), k > 0,$$

where $\hat{d}$ is defined as follows:

$$\hat{d}(x, t) = \begin{cases} \hat{d}_1(t), & x \in \Gamma \cap U(x^{(1)}, \delta(t)), \\ \hat{d}_2(t), & x \in \Gamma \cap (U(x^{(2)}, \delta(t)) \setminus U(x^{(1)}, \delta(t))), \\ \vdots, & x \in \Gamma \cap (U(x^{(i)}, \delta(t)) \setminus \bigcup_{j=1}^{i-1} U(x^{(j)}, \delta(t))), \\ \hat{d}_N(t), & x \in \Gamma \cap (U(x^{N(t)}, \delta(t)) \setminus \bigcup_{j=1}^{N(t)-1} U(x^{(j)}, \delta(t))). \end{cases}$$

By (3.16), we regard $\hat{d}$ as an approximation of $d$ in $L^2(\Gamma)$ (precise convergence is stated in (3.20) later). Under feedback (3.17), the closed-loop system of (1.1) becomes

$$w_t(x, t) = w_t(x, t) \in \Omega, t > 0, \\
\Delta w(x, t) |_{\Gamma} = -k \frac{\partial w_t(x, t)}{\partial \nu} - \hat{d}(x, t) + d(x, t), \\
\dot{y}_i(t) = y_{2i}(t) + \int_{\Gamma} u(x, t)\varphi'_i(x)dx + \hat{d}_i(t) \int_{\Gamma} \varphi'_i(x)dx - r(t)[\hat{y}_i(t) - y_i(t)], \\
\frac{d}{dt} \left( \int_{\Gamma} \varphi'_i(x)dx \right) = -r^2(t)[\hat{y}_i(t) - y_i(t)], i = 1, 2, \ldots.$$

Now we state the main result of this paper.

**Theorem 3.3.** Let $\{x^{(i)}\}$ be as defined in Lemma 3.1. Let $g'_i$, $\hat{a}_i$, $\hat{d}_i$, and $(y_i, y_{2i})$ be as introduced in the equations (3.5), (3.15), (3.18), and (3.13), respectively. Then for any initial value $(w(\cdot, 0), w_t(\cdot, 0))^T \in \mathcal{H}$, the closed-loop system (3.19) admits a unique solution $(w(\cdot, t), w_t(\cdot, t))^T \in C(0, \infty; \mathcal{H})$. In addition, under the conditions (1.2), (1.4), (2.6), (3.1), (3.2), and (3.3), system (3.19) is asymptotically stable in the sense of

$$\lim_{t \to \infty} E_i(t) = 0$$

uniformly for $i = 1, 2, \ldots$, where

$$E_i(t) = \int_{\Omega} |\Delta w(x, t)|^2 + |w_t(x, t)|^2 + \gamma |\nabla w_t(x, t)|^2]dx$$

$$+ |\dot{y}_i(t)| + \int_{\Gamma} |\hat{d}(x, t) - d(x, t)|^2 dx.$$

**Remark 3.1.** In Theorem 3.3, both $d(\cdot, t)$ and $d_t(\cdot, t)$ are supposed to be uniformly bounded for time $t$ in $L^2(\Gamma)$. The boundedness of $d$ is necessary because this ensures
that the control law (3.17) is bounded. This is the basic requirement for ADRC due to
its estimation/cancellation nature and many other methods even sliding mode control
([11, 12, 13]). However, as we indicated in the end of the previous section that the
boundedness of \( d_i(\cdot, t) \) with respect to time \( t \) is not necessary since otherwise, some
disturbances like \( d(x, t) = \sin(x^2 t) \) are excluded. From the proof of Theorem 3.3, we
see that the boundedness of \( d_i(\cdot, t) \) is to guarantee that

\[
\frac{d}{dt} \int_\Gamma d(x, t)\varphi_i'(x)dx = \int_\Gamma d_i(x, t)\varphi_i'(x)dx + \int_\Gamma d(x, t)\varphi_i''(x)dx
\]

is uniformly bounded with respect to \( t \). This is equivalent to the boundedness of the
term on the right-hand side of (3.22). By the construction of \( g_i^s \) in (3.5), this
term can be uniformly bounded provided that

\[
\|d_i(\cdot, t)\|_{L^\infty(\Gamma)} \delta^{n+1}(t)
\]

is uniformly bounded, where \( \delta \) is defined by (3.3). Since \( \lim_{t \to \infty} \delta(t) = 0, \|d_i(\cdot, t)\|_{L^\infty(\Gamma)} \)
can be relaxed to grow slowly than \( \frac{1}{\delta^{n+1}(t)} \). This relaxes the limitation of \( d_i \) in large
extent.

Remark 3.2. In state feedback estimate (3.15), if we replace \( r(t) \) by the constant
high gain \( r(t) \equiv 1/\kappa \), we obtain the following constant high gain estimator:

\[
\begin{cases}
\dot{\tilde{y}}_i(t) = y_{2i}(t) + \int_\Gamma u(x, t)\varphi_i'(x)dx + \dot{d}_i(t) \int_\Gamma \varphi_i'(x)dx - \frac{1}{\kappa}[\tilde{y}_i(t) - y_i(t)], \\
\frac{d}{dt} \left( \hat{d}_i(t) \int_\Gamma \varphi_i'(x)dx \right) = -\frac{1}{\kappa}^2[\tilde{y}_i(t) - y_i(t)], \quad i = 1, 2, \ldots.
\end{cases}
\]

The convergence (3.20) becomes

\[
\lim_{t \to \infty} \sup_i E_i(t) \leq C\kappa,
\]

where \( C > 0 \) is independent of \( \kappa \) and \( i \). The constant high gain (3.24) shares the
advantage of many high gain controls that the high frequency noise can be filtered yet
brings the peaking value problem ([31]). Recommended control strategy is to use the
time varying gain first to reduce the peaking value in the initial stage to a reasonable
level and then apply the constant high gain. This will be explained numerically in
Section 5.

4. Proof of the main results. Proof of Lemma 3.2. Let

\[
\tilde{y}_i(t) = r(t)[\tilde{y}_i(t) - y_i(t)], \quad \hat{d}_i(t) = \hat{d}_i(t) - d(\xi_i(t), t), \quad i = 1, 2, \ldots,
\]

be the errors. By (3.14) and (3.15), \( (\tilde{y}, \hat{d}) \) satisfies

\[
\frac{d}{dt} \left( \tilde{y}_i(t) \int_\Gamma \varphi_i'(x)dx \right) = r(t) \left( \begin{array}{cc}
-1 & 1 \\
-1 & 0
\end{array} \right) \left( \begin{array}{c}
\hat{y}_i(t) \\
\hat{d}_i(t) \int_\Gamma \varphi_i'(x)dx
\end{array} \right) + \left( \frac{r(t)}{\dot{r}(t)} \tilde{y}_i(t) \\
-\frac{d}{dt} (d(\xi_i(t), t) \int_\Gamma \varphi_i'(x)dx) \right), \quad i = 1, 2, \ldots.
\]

We denote by

\[
V_i(t) := \left( \tilde{y}_i(t), \hat{d}_i(t) \int_\Gamma \varphi_i'(x)dx \right) P \left( \tilde{y}_i(t), \hat{d}_i(t) \int_\Gamma \varphi_i'(x)dx \right)^\top, \quad i = 1, 2, \ldots,
\]
where \( \{\tilde{y}_i, \tilde{d}_i\} \) is the solution of (4.2) and the \( 2 \times 2 \) positive definite matrix \( P \) which is different from (2.10) by abuse of notation temporarily in this proof, is the solution of the following Lyapunov equation:

\[
F^T P + PF = -I_{2 \times 2}, \quad F = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Similar to (2.12) to (2.15), we can obtain

\[
\left| \tilde{d}_i(t) \int_{\Gamma} \varphi_i'(x) dx \right| \leq \frac{\sqrt{V_i(t)}}{\sqrt{\lambda_{\min}(P)}} \leq \frac{\sqrt{V_i(t_0)}}{\sqrt{\lambda_{\min}(P)} e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma}} + \frac{N_2}{\lambda_{\min}(P)} e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma} \int_{t_0}^t e^{\int_{t_0}^s \frac{1}{\tau_{\max}(\tau)} r(\tau) d\tau} ds, \quad \forall t \geq t_0, i = 1, 2, \ldots,
\]

where \( N_2 > 0 \) and \( t_0 > 0 \) depend on \( \varepsilon \) only. In particular, (4.4) implies that

\[
\sqrt{V_i(t)} \leq \frac{\sqrt{V_i(t_0)}}{e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma}} + \frac{N_2}{\lambda_{\min}(P)} e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma} \int_{t_0}^t e^{\int_{t_0}^s \frac{1}{\tau_{\max}(\tau)} r(\tau) d\tau} ds, \quad \forall t \geq t_0, i = 1, 2, \ldots
\]

By (3.6), (3.11), (3.13), and (3.15), we may suppose that \( |\sqrt{V_i(t_0)}| \leq C \) for all \( i \geq 1 \) for some \( i \)-independent constant \( C > 0 \). Applying the L’Hospital rule and assumption (2.6) to the second term of the right-hand side of (4.5), and assumption (2.6) to the first term, we obtain

\[
\lim_{t \to \infty} V_i(t) = 0
\]

uniformly for \( i = 1, 2, \ldots \). Therefore,

\[
\lim_{t \to \infty} |\tilde{y}_i(t) - y_i(t)| = 0 \text{ uniformly for } i = 1, 2, \ldots.
\]

Since

\[
\frac{1}{2^{n-1} C_1(\Gamma)} \Pi_{n-1} \delta^{n+1}(t) \leq \int_{\Gamma} \varphi_i'(x) dx \leq C_1(\Gamma) \Pi_{n-1} \delta^{n+1}(t),
\]

where \( \Pi_{n-1} \) is the unit volume of \( \mathbb{R}^{n-1} \) and \( C_1(\Gamma) \) is a positive constant which depends on \( \Gamma \) only, we have, for all \( t \geq t_0 \) and \( i = 1, 2, \ldots \), that

\[
|\tilde{d}_i(t)| \leq \frac{2^{n-1} C_1(\Gamma)}{\Pi_{n-1}} \left( \frac{C}{\sqrt{\lambda_{\min}(P)} \delta^{n+1}(t) e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma}} + \frac{N_2}{\lambda_{\min}(P)} \delta^{n+1}(t) e^{\int_{t_0}^t \frac{1}{\tau_{\max}(\sigma)} r(\sigma) d\sigma} \right)
\]

We claim that \( \tilde{d}_i(t) \to 0 \) as \( t \to \infty \). To this purpose, it suffices to show the convergence of the second term of the right-hand side of (4.9) since the first term is less than the
second term up to a constant as \( t \to \infty \). Using the L’Hospital rule and assumption (3.3), we obtain

\[
\lim_{t \to \infty} \frac{\int_{t_n}^t e^{\int_{t_0}^s \frac{1}{\lambda_{\max}(P)} r(\sigma) d\sigma} d\sigma}{\delta^{n+1}(t)} = \lim_{t \to \infty} \frac{e^{\int_{t_0}^t \frac{1}{\lambda_{\max}(P)} r(\sigma) d\sigma}}{\delta^{n+1}(t)} = \frac{1}{(n+1)\delta(t)}.
\]

This, together with (4.9), proves (3.16).

The proof of Theorem 3.3. Using the error variables \((\hat{y}_i, \hat{d}_i)\) defined in (4.1), we can write the equivalent form of system (3.19) as follows:

\[
\begin{align*}
\dot{w}_t(x, t) - \gamma \Delta w_t(x, t) + \Delta^2 w(x, t) &= 0, x \in \Omega, t > 0, \\
\langle w(x, t) \rangle_r &= 0, \\
\Delta w(x, t)|_{\Gamma} &= -k \frac{\partial w_t(x, t)}{\partial n} - \hat{d}(x, t) + d(x, t), \\
\hat{y}_i(t) &= -r(t)\hat{y}_i(t) + r(t)\hat{d}_i(t) - \int_{\Gamma} \varphi_i'(x) dx + \frac{\dot{r}(t)}{r(t)} \hat{y}_i(t), \\
\frac{d}{dt} \left( \hat{d}_i(t) \int_{\Gamma} \varphi_i'(x) dx \right) &= -r(t)\hat{y}_i(t) - \frac{d}{dt} \left( d(\xi_i(t), t) \int_{\Gamma} \varphi_i'(x) dx \right),
\end{align*}
\]

The “ODE part” of (4.10) has been actually proven tending to zero as \( t \to \infty \) in (4.7) and (3.16). More precisely, let \( \tilde{d}(x, t) = \hat{d}(x, t) - d(x, t) \). One can show that

\[
\lim_{t \to \infty} \| \tilde{d}(\cdot, t) \|_{L^2(\Gamma)} = 0.
\]

Indeed, since \( d(\cdot, t) \) satisfies \( |d(x', t) - d(x'', t)| \leq K(t)|x' - x''|^\alpha \), for the given \( \varepsilon(t) > 0 \), since \( \delta(t) = \left( \frac{\varepsilon(t)}{K(t)} \right)^{\frac{1}{\alpha}} > 0 \), we have \( |d(x', t) - d(x'', t)| \leq 2^\alpha \varepsilon(t) \) as long as \( |x' - x''| \leq 2\delta(t) \). Moreover, since \( \xi_i : [0, \infty) \to \Gamma \cap U(x^{(i)} \delta(t)) \) which is defined in (3.11), we have

\[
\| d(\xi_i(t), t) - d(\cdot, t) \|_{L^2(\Gamma \cap U(x^{(i)} \delta(t)))} \leq \text{meas}(\Gamma \cap U(x^{(i)} \delta(t)))2^\alpha \varepsilon(t), \forall t \geq 0.
\]
This together with (4.9) yields
\[
\| \tilde{d}(\cdot, t) \|_{L^2(\Gamma)} \leq \sum_{i=1}^{N(t)} \left[ \| \tilde{d}(\cdot, t) - d(\xi_i(\cdot), t) \|_{L^2(\Gamma \cap U(x_i, \delta(t)))} \right]
\]
\[
+ \| d(\xi_i(\cdot), t) - d(\cdot, t) \|_{L^2(\Gamma \cap U(x_i, \delta(t)))}
\]
\[
\leq \text{meas}(\Gamma)^{2n-1}C_1(\Gamma) \left( \frac{C}{\Pi_1(\Gamma)^{\delta n + 1(t)}e^{\int_0^t \frac{1}{\max(r(\sigma))} d\sigma}} \right)
\]
\[
+ \frac{N_2}{\delta n + 1(t)} \int_{\Gamma} e^{\int_0^t \frac{1}{\max(r(\sigma))} d\sigma} d\sigma
\]
\[
+ C^n(\Gamma)(n - 1)\frac{\delta n + 1(t)}{\delta n + 1(\sigma)} \text{meas}(\Gamma)\eta(t), \forall t \geq t_0,
\]
where we used Lemma 3.1 and (4.9). Since \( \lim_{t \to \infty} \eta(t) = 0 \), applying the L'Hospital rule and assumption (3.3) to (4.13) leads to (4.11). Now, we consider the “w part” of system (4.10) which is rewritten as
\[
\begin{cases}
    w_t(x, t) - \gamma \Delta w_t(x, t) + \Delta^2 w(x, t) = 0, \\
    w(x, t)|_{\Gamma} = 0, \\
    \Delta w(x, t)|_{\Gamma} = -k \frac{\partial w_t(x, t)}{\partial \nu} - \tilde{d}(x, t) + d(x, t).
\end{cases}
\]
Equation (4.14) is similar to the “w part” of system (2.19). Taking (4.11) into account and using the same arguments from (2.21) to (2.28), we can prove that the solution of (4.14) satisfies
\[
\lim_{t \to \infty} \left\| \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} \right\|_{H} = \lim_{t \to \infty} \int_{\Omega} [\Delta^2 w(x, t)]^2 + |w_t(x, t)|^2 + \gamma |\nabla w_t(x, t)|^2] dx = 0.
\]
Finally, it follows from (4.1) that
\[
\begin{cases}
    \tilde{y}_i(t) = \frac{\tilde{y}_i(t)}{r(t)} + y_i(t), \\
    y_i(t) = \int_{\Omega} [w_t(x, t)g_i^t(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^t(x)] dx, \; i = 1, 2, \ldots.
\end{cases}
\]
By (4.7) and assumption (2.6),
\[
\frac{\tilde{y}_i(t)}{r(t)} \to 0 \; \text{as} \; t \to \infty
\]
uniformly for \( i \). Since \( \| g_i^t \|_{L^2(\Omega)} \leq C \| \nabla g_i^t \|_{L^2(\Omega)} \leq C \| g_i^t \|_{H^1(\Omega)} \leq C \| \nabla g_i^t(x) \|_{H^{\frac{1}{2}}(\Gamma)} \leq C\| \nabla g_i^t(x) \|_{H^{\frac{1}{2}}(\Gamma)} \leq C\delta^{n-2}(t) \leq C\delta^{n-2}(0) \) for all \( i \) with some constant \( C > 0 \) by construction (3.6) (see the elliptic estimation in [24, p.188] and the Sobolev embedding theorem in [1, p.97]), it follows from (4.15) that
\[
\int_{\Omega} [w_t(x, t)g_i^t(x) + \gamma \nabla w_t(x, t) \cdot \nabla g_i^t(x)] dx \to 0 \; \text{as} \; t \to \infty
\]
uniformly for $i$. Combining (4.17) and (4.18) gives
\[ \hat{y}_i(t) \to 0 \text{ as } t \to \infty \]
uniformly for $i$. This ends the proof. \hfill \Box

**Proof of Remark 3.2.** We only give a sketch of the proof. From the proof of Theorem 3.3, we see that to get (3.24), it suffices to show that
\[ \text{uniformly for } t \geq t_0. \]

For $\tilde{d}$ defined by (4.1), similar to equations from (2.12) to (2.15), one can get the following estimation that is subtler than estimations (4.4) and (4.9):
\[ |\tilde{d}(t)| \leq C \frac{\Pi_{n-1}}{\Pi_{n+1}} \left( \frac{C}{\sqrt{\text{max}(P)}} \right)^{\delta n + 1 \delta(n+1)} \]
\[ \text{(4.20)} \]
\[ + \frac{N_2}{\text{max}(P)} \int_{t_0}^{t} \frac{d(\xi_i(s), \int_{\Gamma} \varphi_i(x) dx)}{\text{max}(P)} ds \]
\[ \text{(4.22)} \]

By condition (3.2), we can choose $\delta(t) = \frac{1}{1+t}$. Since
\[ (1+t)^{n+1} e^{-\frac{t-t_0}{\text{max}(P)}} \leq C_1 \kappa, \quad \forall \, t \geq t_0, \]
for some constant $C_1 > 0$ independent of $\kappa$, it follows from (1.2), (3.7), and (3.11) that
\[ \lim_{t \to \infty} \sup_{t \geq t_0} \int_{\Gamma} \frac{d(\xi_i(s), \int_{\Gamma} \varphi_i(x) dx)}{\text{max}(P)} ds \]
\[ = \lim_{t \to \infty} \sup_{t \geq t_0} \frac{\frac{d(\xi_i(s), \int_{\Gamma} \varphi_i(x) dx)}{\text{max}(P)} ds}{(n+1)\delta^n(\delta(t) + \frac{\delta^{n+1}(t)}{\text{max}(P)\kappa})} \]
\[ \leq C_2 \max\{\|d\|_{L^\infty(0,\infty;C(\Gamma))}, \|d\|_{L^\infty(0,\infty;C(\Gamma))}\} \]
\[ \times \lim_{t \to \infty} \sup_{t \geq t_0} \frac{\delta^{n+1}(t) + \delta^n(\delta(t) + \frac{\delta^{n+1}(t)}{\text{max}(P)\kappa})}{(n+1)\delta^n(\delta(t) + \frac{\delta^{n+1}(t)}{\text{max}(P)\kappa})} \]
\[ = C_2 \max\{\|d\|_{L^\infty(0,\infty;C(\Gamma))}, \|d\|_{L^\infty(0,\infty;C(\Gamma))}\} \]
\[ \times \lim_{t \to \infty} \sup_{t \geq t_0} \frac{1 + \frac{\delta(t)}{\text{max}(P)\kappa}}{(n+1)\delta^n(\delta(t) + \frac{\delta^{n+1}(t)}{\text{max}(P)\kappa})} \]
\[ \leq 4\lambda_{\text{max}}(P)C_2 \max\{\|d\|_{L^\infty(0,\infty;C(\Gamma))}, \|d\|_{L^\infty(0,\infty;C(\Gamma))}\}\kappa, \]
where $C_2 > 0$ is independent of $\kappa$ and $d$. Combining (4.20), (4.21), and (4.22) gives
\[ \lim_{t \to \infty} \sup_{t \geq t_0} |\tilde{d}(t)| \leq C \max\{\|d\|_{L^\infty(0,\infty;C(\Gamma))}, \|d\|_{L^\infty(0,\infty;C(\Gamma))}\}\kappa. \]

This gives (4.19).
5. Numerical simulations. In this section, we present some numerical simulations for illustration. The purpose is twofold. The first is to verify the theoretical result and the second is to look at the peaking value reduction by the time varying gain. For computational simplicity, we just take dimension \( n = 2 \) and the disturbance to be varying with respect to both time and space. The system is presented in Example 1.1 in Section 1 already but here we re-write the closed-loop form (3.19) for the sake of clarity.

\[
\begin{align*}
\begin{cases}
    w_{tt}(x,t) - \gamma \Delta w_{tt}(x,t) + \Delta^2 w(x,t) = 0, x \in \Omega, t \geq 0, \\
    w(x,t)|_{\Gamma} = 0, \\
    \Delta w(x,t)|_{\Gamma} = -k \frac{\partial w_t(x,t)}{\partial \nu} - \hat{d}(x,t) + d(x,t), \\
    \dot{y}_i(t) = y_{2i}(t) + \int_{\Gamma} u(x,t) \phi^i_1(x) dx + \hat{d}_i(t) \int_{\Gamma} \phi^i_2(x) dx - r(t) \hat{y}_i(t) - y_i(t), \\
    \frac{d}{dt} \left( \hat{d}_i(t) \int_{\Gamma} \phi^i_1(x) dx \right) = -r^2(t) \hat{y}_i(t) - y_i(t), \quad i = 1, 2, \ldots.
\end{cases}
\end{align*}
\]

where \( \gamma = 0.001, \Omega = \{ x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 1 \}, \Gamma = \{ x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1 \}, \hat{d} \) is defined by (3.18), \( y_i \) and \( y_{2i} \) are defined by (3.13), \( g^i_1 \) is defined by (3.6), and \( \phi^i_2 \) is by (3.7). For numerical computations, we take parameter \( k = 3, d(x,t) = \sin(x_2 t) \), and the initial values as follows:

\[
\begin{align*}
\begin{cases}
    w(x_1, x_2, 0) = 5 \sin \left( 6 \sqrt{x_1^2 + x_2^2} - 6 \right) \left( \frac{4x_1^2}{(x_1^2 + x_2^2)^{1.5}} - \frac{3x_1}{\sqrt{x_1^2 + x_2^2}} \right), \\
    w_t(x_1, x_2, 0) = 4(x_1^2 + x_2^2 - 1) \left( \frac{3x_2}{\sqrt{x_1^2 + x_2^2}} - \frac{4x_1^2}{(x_1^2 + x_2^2)^{1.5}} \right), \\
    \dot{y}_i(0) = y_i(0) - 10, \quad \dot{d}_i(0) = d_i(0) + 2, i = 1, 2, \ldots.
\end{cases}
\end{align*}
\]

Since the spatial domain consists of a two-dimensional disk, we can transform the disk into rectangle under the polar coordinate so that the numerical computation is easily performed in rectangle region. We first solve (5.1) under the polar coordinate and then convert back to the original coordinate for some figures if necessary. Under
we plot initial time for both varying gain is slightly slower than the convergence with the constant gain which is convergence is fast and satisfactorily. The price is that the convergence with the time the constant gain (Figures 5.2(b) and 5.3(b)). It is clearly seen that in both cases, the and 5.3, respectively, with both the time varying gain (Figures 5.2(a) and 5.3(a)) and 

The numerical algorithm is programmed by Matlab ([32]) and the numerical results are plotted in Figures 5.2-5.5. 

The backward Euler method in time and Chebyshev spectral method for polar variables are used to discretize system (5.3). Here we take the grid size \( \rho_N = 16 \) for \( \rho \), the grid size \( \theta_N = 24 \) for \( \theta \), and the time step \( dt = 2 \times 10^{-4} \). The time varying gain function \( r \) is taken as 

\[
r(t) = \begin{cases} 
e^{5t}, & t \leq \frac{\log(30)}{5}, \\ 30, & t > \frac{\log(30)}{5}. \end{cases}
\]

It is seen that \( r \) grows slowly from the small value in the beginning to its maximum value \( r = 30 \) which is used as the constant high gain in our numerical simulations. The numerical algorithm is programmed by Matlab ([32]) and the numerical results are plotted in Figures 5.2-5.5. 

Figures 5.1(a) and 5.1(b) display the displacement \( w \) and the velocity \( w_1 \) at the initial time \( t = 0 \) and the time \( t = 80 \), respectively. It is clearly seen for both \( w \) and \( w_1 \). 

To compare the effects of the time varying gain (5.5) and the constant gain \( r = 30 \), we plot \( w(\rho, \pi/2, t) \) and \( w_1(\rho, \pi/2, t) \) in polar coordinates for system (5.3) in Figures 5.2 and 5.3, respectively, with both the time varying gain (Figures 5.2(a) and 5.3(a)) and the constant gain (Figures 5.2(b) and 5.3(b)). It is clearly seen that in both cases, the convergence is fast and satisfactorily. The price is that the convergence with the time varying gain is slightly slower than the convergence with the constant gain which is
not very clear from Figure 5.2 but clear from Figure 5.3. This is also observed in the succeeding Figures 5.4 and 5.5 for the disturbance tracking.

Figure 5.4 plots the tracking errors for the disturbance where Figure 5.4(a) is with the time varying gain and Figure 5.4(b) is with the constant gain $r = 30$. It is clearly seen from these figures that the peaking value from Figure 5.4(b) is dramatically reduced by the time varying gain in Figure 5.4(a). This is the biggest advantage of using time varying gains instead of constant gains as is common in existing literature [11, 12, 13]. This is also a remarkable property of ADRC in dealing with disturbance. Many engineering examples show that we actually do not need much high gain for the convergence due to the nature of estimation/cancellation in ADRC although it is difficult to prove this fact theoretically. The convergence and peaking reduction are also clearly observed from the specific direction $\theta = \pi/2$ under the polar coordinate in Figure 5.5 where Figure 5.5(a) is with the time varying gain and Figure 5.5(b) is with the constant gain.

Finally, we indicate the robustness of our algorithm to arbitrarily small noise. If we continuously apply the increasing time varying gain, it brings bad robustness to high frequency noise. However, this is not the practical case. As we have mentioned at the end of Section 3 that Remark 3.2 suggests us to apply the time varying gain first to reduce the peaking value in the initial stage to a reasonable level and then apply the constant high gain. This is just what we have done by time varying gain (5.5). In this way, the high frequency can be effectively filtered. In Figure 5.6, we plot the displacement of $w(p, \frac{\pi}{2}, t)$ under the polar coordinate by time varying gain (5.5) with different pure noises $d(t) = 200 \sin(10t); d(t) = 200 \sin(50t); d(t) = 200 \sin(300t)$, respectively. It is seen that in the fixed time interval, the higher the frequency in noise, the less affection to the state output.

6. Concluding remarks. In this paper, we apply the active disturbance rejection control approach to boundary state feedback stabilization for a multi-dimensional Kirchoff equation suffered from the boundary disturbance. It is shown that for a quite general boundary disturbance no matter it depends on time only or it depends both time and spatial variable, a state feedback estimator can estimate effectively the disturbance in real time. After recovering the disturbance from the state feedback estimator, the disturbance is canceled in the feedback loop. The collocated feedback control is then applied to stabilize the overall system. The advantages of complete
The displacement \( w(\rho, \frac{\pi}{2}, t) \) with the time varying gain

The displacement \( w(\rho, \frac{\pi}{2}, t) \) with the constant gain

Fig. 5.2. The evolution of \( w(\rho, \frac{\pi}{2}, t) \) under the polar coordinate with both time varying gain and constant gain (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this article).

The velocity \( w_t(\rho, \frac{\pi}{2}, t) \) with time-varying gain

The velocity \( w_t(\rho, \frac{\pi}{2}, t) \) with constant gain

Fig. 5.3. The evolution of \( w_t(\rho, \frac{\pi}{2}, t) \) under the polar coordinate with both time varying gain and constant gain (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this article).

The error \( \hat{d} - d \) under the time varying high gain; (b) The error \( \hat{d} - d \) under the constant high gain (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this article).

Fig. 5.4. (a) The error \( d - \hat{d} \) under the time varying gain; (b) The error \( d - \hat{d} \) under the constant gain (for interpretation of the references to color of the figure’s legend in this section, we refer to the PDF version of this article).

Disturbance rejection are presented both theoretically and numerically. In particular, the numerical experiments show that the peaking value problem caused by the constant high gain in the state feedback estimator can be dramatically reduced through the time varying gain. The price paid by this type of time varying gain is that it takes
a slightly longer time to track the true value of the disturbance and bad robustness to high frequency noise compared with the constant gain. The suggested strategy is to apply the time varying gain first to reduce the peaking value in the initial stage to a reasonable level and then apply the constant high gain. This strategy can reduce the peaking value and filter the high frequency noise simultaneously, which is vali-
dated by numerical experiment. Finally, we point out that our feedback is the full state feedback which serves as a first step for designing the output feedback in future investigations. An output feedback for a one-dimensional PDE with disturbance by sliding mode control is reported in [2].

REFERENCES

[23] I. Lasiecka and R. Triggiani, Exact controllability and uniform stabilization of Kirchhoff plates with boundary control only on ∆ω|Σ; and homogeneous boundary displacement, J.


7. Appendix. Proof of Lemma 3.1. Since $\Omega \subset \mathbb{R}^n$ is an open bounded set and its boundary $\Gamma$ is of $C^4$-class, for any $x^0 = (x_1, x_2, \ldots, x_n) \in \Gamma$, there exists a $C^4$-function $\psi$ and a neighborhood $U_{\epsilon, \rho} \subset \Gamma$ of $x^0$ such that $x_j = \psi(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1})$ holds in $U_{\epsilon, \rho}$ for some $j$ (see, e.g., [7, p.626]). Since $\Gamma$ is compact in $\mathbb{R}^{n-1}$, by finite covering theorem, we can decompose $\Gamma$ as $\Gamma = \bigcup_{i=1}^{p} \Gamma^i$ for some integer $p \geq 1$ such that each $\Gamma^i$ is open in $\Gamma$ and $\Gamma^i$ can be expressed, for some $j$, as

$$x_j = \psi_i(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}), \forall (x_1, x_2, \ldots, x_{n-1}) \in \Gamma^i,$$

where $\psi_i \in C^4$. Therefore, we may assume without loss of generality that $\Gamma$ can be described by $x_n = \psi(x_1, x_2, \ldots, x_{n-1}) \in C^4(\Omega_{n-1})$ for some hypercube $\Omega_{n-1} \subset \mathbb{R}^{n-1}$.

Let $\Omega_c$ be a $(n-1)$-hypercube in $\mathbb{R}^{n-1}$ space. Suppose that each side of $\Omega_c$ parallels the corresponding orthogonal coordinate axis of $\mathbb{R}^{n-1}$ so that $\Omega_{n-1} \subset \Omega_c$. Let

$$C(\Gamma) = \left\| \sqrt{1 + \psi_1^2 + \psi_2^2 + \ldots + \psi_{n-1}^2} \right\|_{C(\Omega_{n-1})}, \quad \delta_i(t) = \frac{\delta_i(t)}{C(\Gamma)}.$$ (7.1)

We suppose that $\Omega_c = \bigcup_{j=1}^{k_0} U_{\text{rect}}(y_0, \frac{\delta_i(0)}{\sqrt{n-1}})$ and $U_{\text{rect}}(y_0, \frac{\delta_i(0)}{\sqrt{n-1}}) \cap U_{\text{rect}}(y_0, \frac{\delta_i(0)}{\sqrt{n-1}}) = \emptyset$ for $1 \leq p \neq q \leq k_0 \geq 1$, where $U_{\text{rect}}(y^*, r) = \{ y \in \mathbb{R}^{n-1} \mid |y_i - y_{i}^*| < r, i = 1, 2, \ldots, n-1 \}$ denotes a hypercube of $\mathbb{R}^{n-1}$ where $y_i$ is the $i$-th component of $y$ and so is $y_{i}^*$.

We first state a simple fact on geometry of $\mathbb{R}^{n-1}$. For $\rho > 0$, let $S = \{ z_h = (\rho_1, \ldots, \rho_{n-1}) \mid |i| \in \mathbb{Z}, j = 1, 2, \ldots, n-1 \}$ be the set of grid points of $\mathbb{R}^{n-1}$. Let $\{ U_{\text{rect}}(z_h, \rho) \mid z_h \in S \}$ be a set of hypercube of $\mathbb{R}^{n-1}$. Then, the length of boundary of $U_{\text{rect}}(z_h, \rho)$ along any parallel direction of the orthogonal coordinate axis of $\mathbb{R}^{n-1}$ is just $2\rho$. It is seen that along a fixed coordinate parallel direction of $\mathbb{R}^{n-1}$, any
interior point of $U_{\text{rect}}(z_h, \rho)$ belongs to at most two hypercubes of $\{U_{\text{rect}}(z_h, \rho)\}$. Since all interior points of $\{U_{\text{rect}}(z_h, \rho)\}$ are just $\mathbb{R}^{n-1}$, we conclude that any point of $\mathbb{R}^{n-1}$ belongs to at most two hypercubes of $\{U_{\text{rect}}(z_h, \rho)\}$. Since $\mathbb{R}^{n-1}$ has $n-1$ number of axes, any point of $\mathbb{R}^{n-1}$ belongs to at most $2^{n-1}$ number of hypercubes of $\{U_{\text{rect}}(z_h, \rho)\}$.

Let $\mathcal{X}(0) = \{y_0^{(j)}\mid U_{\text{rect}}(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}}) \cap \Omega_{n-1} \neq \emptyset\}$. Obviously, $\{U_{\text{rect}}(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}})\mid y_0^{(j)} \in \mathcal{X}(0)\}$ is a hypercube cover of $\Omega_{n-1}$. Taking $\rho = \frac{\delta_1(0)}{\sqrt{n-1}}$ as that in above paragraph, we see that there are at most $2^{n-1}$ number of hypercubes such that

$$
y \in \bigcap_{j=1}^{2^{n-1}} U_{\text{rect}}\left(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}}\right), y_0^{(j)} \in \mathcal{X}(0), \forall y \in \Omega_{n-1}
$$

and hence

$$
\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{\text{rect}}\left(y_0^{(j)}, \frac{\delta_1(0)}{\sqrt{n-1}}\right)\right) \leq 2^{n-1} \text{meas}(\Omega_{n-1}),
$$

where $N(0) = \#\mathcal{X}(0)$. By the continuity and monotonicity of $\delta(t)$, for sufficiently small $t > 0$,

$$
\sum_{j=1}^{N(0)} \text{meas}\left(\Omega_{n-1} \cap U_{\text{rect}}\left(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right) \leq 2^{n-1} \text{meas}(\Omega_{n-1}).
$$

Since $\lim_{t \to 0} \delta_1(t) = 0$, there exists a $t^* > 0$ such that for all $t > t^*$, $\{\Omega_{n-1} \cap U_{\text{rect}}(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}})\}_{j=1}^{N(0)}$ cannot cover $\Omega_{n-1}$. Let

$$
t_1 = \inf \left\{t > 0 \mid \Omega_{n-1} \cap U_{\text{rect}}\left(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right) \text{ cannot cover } \Omega_{n-1} \right\}.
$$

Note that the boundary of each $U_{\text{rect}}(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}})$ ($0 \leq j \leq N(0)$) consists of $2(n-1)$ number of boundary hypercubes $\{U_{\text{rect}}^{j,k}\}_{k=1}^{2(n-1)} \subset \mathbb{R}^{n-2}$. Along any oriented coordinate axis direction of $\mathbb{R}^{n-2}$, the length of each hypercube $U_{\text{rect}}^{j,k}$ is $\frac{\delta_1(t)}{\sqrt{n-1}}$. We partition symmetrically each hypercube $U_{\text{rect}}^{j,k}$ into $2^{n-2}$ number of hypercubes $\{U_{\text{rect}}^{j,k,l}\}_{l=1}^{2^{n-2}} \subset \mathbb{R}^{n-2}$ so that the boundary length of each $U_{\text{rect}}^{j,k,l}$ along any oriented coordinate axis direction of $\mathbb{R}^{n-2}$ is just $\frac{\delta_1(t)}{\sqrt{n-1}}$. Choose $\{y_l^{(j)}\}_{j=N(0)+1}^{N(1)}(t)$ as all vertexes of all $U_{\text{rect}}^{j,k,l}$ for $0 \leq j \leq N(0)$, $1 \leq k \leq 2(n-2)$, $1 \leq l \leq 2^{n-2}$. These vertexes are considered as points of $\mathbb{R}^{n-1}$.

In this way, we have

$$
\Omega_{n-1} \subset \left\{U_{\text{rect}}\left(y_1^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right\}_{j=N(0)+1}^{N(1)} \bigcup \left\{U_{\text{rect}}\left(y_0^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right)\right\}_{j=1}^{N(0)}.
$$

For notation simplicity, we still denote by $\{y^{(j)}\}_{j=1}^{N(1)}(t) = \{y_l^{(j)}\}_{j=N(0)+1}^{N(1)}, \{y^{(j)}\}_{j=1}^{N(0)}(0) = \{y_l^{(j)}\}_{j=1}^{N(0)}$.

Same to (7.2) and (7.3) with $\rho = \frac{\delta_1(t)}{\sqrt{n-1}}$, we see that there are at most $2^{n-1}$ number of hypercubes such that

$$
y \in \bigcap_{j=1}^{2^{n-1}} U_{\text{rect}}\left(y^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}}\right), y^{(j)} \in \mathcal{X}(t_1), \forall y \in \Omega_{n-1}
$$
and
\begin{equation}
\sum_{j=1}^{N(t_1)} \text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right) \leq 2^{n-1} \text{meas}(\Omega_{n-1}),
\end{equation}

where $\mathcal{X}(t_1) = \mathcal{X}(0) \cup \{y^{(j)}\}_{j=N(0)+1}^{N(t_1)}$, $N(t_1) = \#\mathcal{X}(t_1)$. By induction, there exist $\{t_i\}_{i=2}^{\infty}$ and $\{y^{(j)}\}_{j=1}^{N(t_i)}$ such that
\begin{equation}
\forall y \in \Omega_{n-1}, \quad y \in \bigcap_{j=1}^{2^{n-1}} U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t_i)}{\sqrt{n-1}} \right), \quad X(t_i) = X(0) \cup \{y^{(j)}\}
\end{equation}

and
\begin{equation}
\sum_{j=1}^{N(t_i)} \text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t_i)}{\sqrt{n-1}} \right) \right) \leq 2^{n-1} \text{meas}(\Omega_{n-1}),
\end{equation}

where $\mathcal{X}(t_i)$ is defined iteratively by
\begin{equation}
\mathcal{X}(t_{i+1}) = \mathcal{X}(t_i) \cup \{y^{(j)}\}_{j=N(t_i)+1}^{N(t_{i+1})},
\end{equation}

By this construction, we see that the bounded measure cover
\begin{equation}
\bigcup_{j=1}^{N(t_i)} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t_i)}{\sqrt{n-1}} \right) \right) = \Omega_{n-1}
\end{equation}

is a discrete series of cover which is independent of time $t$. Now we relate this cover with time $t$ by setting
\begin{equation}
\mathcal{X}(t) := \mathcal{X}(t_i), t \in [t_i, t_{i+1}), N(t) = \#\mathcal{X}(t), \lim_{t \to \infty} N(t) = \infty, i = 0, 1, 2, \ldots
\end{equation}

Then we get from (7.8) that for all $t \geq 0$,
\begin{equation}
\bigcup_{j=1}^{N(t)} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right) = \Omega_{n-1},
\end{equation}

and
\begin{equation}
\sum_{j=1}^{N(t)} \text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(j)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right) \leq 2^{n-1} \text{meas}(\Omega_{n-1}).
\end{equation}

Let $x^{(i)} = (y^{(i)}, \psi(y^{(i)})) \in \Gamma$ for $i = 1, 2, \ldots$ Since by (7.1), $\delta(t) = C(\Gamma)\delta_1(t)$, it follows from (7.12) that
\begin{equation}
x \in \bigcup_{i=1}^{N(t)} U \left( x^{(i)}, \delta(t) \right), \forall x \in \Gamma, t \geq 0,
\end{equation}
and by (7.13),

\[
\sum_{i=1}^{N(t)} \text{meas}(\Gamma \cap U(x^{(i)}, \delta(t))) \\
= \sum_{i=1}^{N(t)} \int_{\Omega_{n-1} \cap U((y^{(i)}, 0), \delta(t))} \sqrt{1 + \psi_{x_1}^2 + \psi_{x_2}^2 + \ldots + \psi_{x_{n-1}}^2} \, dx_1 \ldots dx_{n-1} \\
\leq \sum_{i=1}^{N(t)} \int_{\Omega_{n-1} \cap U_{\text{rect}}(y^{(i)}, \delta(t))} \sqrt{1 + \psi_{x_1}^2 + \psi_{x_2}^2 + \ldots + \psi_{x_{n-1}}^2} \, dx_1 \ldots dx_{n-1} \\
\leq \sum_{i=1}^{N(t)} C(\Gamma) \text{meas}(\Omega_{n-1} \cap U_{\text{rect}}(y^{(i)}, \delta(t))) \\
\leq \sum_{i=1}^{N(t)} C^n(\Gamma)(n-1)^{\frac{n-1}{2}} \text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(i)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right) \\
\leq C^n(\Gamma)(n-1)^{\frac{n-1}{2}} 2^{n-1} \text{meas}(\Omega_{n-1}) \\
\leq C^n(\Gamma)(n-1)^{\frac{n-1}{2}} 2^{n-1} \int_{\Omega_{n-1}} \sqrt{1 + \psi_{x_1}^2 + \psi_{x_2}^2 + \ldots + \psi_{x_{n-1}}^2} \, dx_1 \ldots dx_{n-1} \\
= C^n(\Gamma)(n-1)^{\frac{n-1}{2}} 2^{n-1} \text{meas}(\Gamma), \forall t \geq 0.
\]

In the derivation of (7.15), we used a trivial fact in the space \( \mathbb{R}^{n-1} \) that

\[
\text{meas} \left( U_{\text{rect}}(y^{(i)}, \delta(t)) \right) = C^{n-1}(\Gamma)(n-1)^{\frac{n-1}{2}} \text{meas} \left( U_{\text{rect}} \left( y^{(i)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right)
\]

and hence

\[
\text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}}(y^{(i)}, \delta(t)) \right) \leq C^{n-1}(\Gamma)(n-1)^{\frac{n-1}{2}} \text{meas} \left( \Omega_{n-1} \cap U_{\text{rect}} \left( y^{(i)}, \frac{\delta_1(t)}{\sqrt{n-1}} \right) \right)
\]

since \( \Omega_{n-1} \) is supposed to be a hypercube of \( \mathbb{R}^{n-1} \). Another fact that we used in the derivation of (7.15) is that for a \((n-1)\)-dimensional surface \( S = \{ \psi(U) \}, U \subset \mathbb{R}^{n-1}, \psi(x) = (x', \psi^0(x')), x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \psi^0 \in C^1(\mathbb{R}^{n-1}) \),

\[
\text{meas}(S) = \int_U \sqrt{1 + |D\psi^0|^2} \, dx_1 \ldots dx_{n-1},
\]

where \( D\psi^0 = (\psi_{x_1}^0, \psi_{x_2}^0, \ldots, \psi_{x_{n-1}}^0) \) ([5, p.101-102]). Combining (7.14) and (7.15) gives the required result. \( \blacksquare \)