Applied Mathematical Sciences, Vol. 3, 2009, no. 10, 491 - 508

A Stability Study of the Active Disturbance Rejection Control Problem by a Singular Perturbation Approach

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Abstract. We study the stability characteristic of the active disturbance rejection control for a nonlinear, time-varying plant. To this end, the closed-loop system is reformulated in a form that allows the singular perturbation method to be applied. Since singular perturbation approach enables the decomposition of the original system into a relatively slow subsystem and a relatively fast subsystem, the composite Lyapunov function method is used to determine the stability properties of the decomposed subsystems. Our result shows that the system is exponentially stable, upon which a lower bound for the observer bandwidth is established.

Mathematics Subject Classification: 34H05, 34D15, 93C95, 93C10

Keywords: active disturbance rejection control, singular perturbation, exponential stability

1. Introduction

Most control systems will unavoidably encounter disturbances, both internal (pertaining to unknown, nonlinear, time-varying plant dynamics) and external, and the system performance largely depends on how effectively the control system can deal with them. Thus, one of the original and fundamental research topic in control theory is to study the problem of disturbances rejection. In regard to external disturbances, it is well known that good disturbance rejection is achieved with a high gain loop together with a high bandwidth, assuming that the plant is linear, time-invariant (LTI) and accurately described in a mathematical model. Things get a little complicated and interesting when such assumptions do not hold, as in most practical applications, where it is the internal disturbance that is most significant. In many regards, much of the literature on control can be seen as various responses to this dilemma. That is, *how do we take a well-formulated and time-tested, classical control theory and apply it to nonlinear, time-varying (NTV) and uncertain plants in the real world?*

To this end, two technology upgrades, figuratively speaking, are offered in the modern control framework: adaptive control ([1]) and robust control theory ([31]). The former refers to a class of controllers whose gains are adjusted using a particular adaptation law to cope with the unknowns and changes in the plant dynamics; while the latter is based on the optimal control solution, assuming that the plant dynamics is mostly known and LTI, with the given bound on the uncertainties in frequency domain. The combination of the two offers the third alternative: robust adaptive control ([14]). While these proposed solutions show awareness of the problem and progress toward solving it, they are far from resolving it because, among other reasons, they still rely on accurate and detailed mathematical model of the plant and they often produce solutions, such as the H_∞ design, that are rather conservative. Therefore, as improvements of, rather than departure from, the model-based design paradigm, adaptive control and robust control, as solutions to the disturbance rejection problem, didn't travel very far from their source: model-based classical control theory. Such approaches are deemed *passive* as they accept disturbances as they are and merely deal with them as one of the design issues.

In contrast, from the 70s in the last century to the present, there have been many researchers, although scattered and overlooked for the most part, who suggested various approaches to disturbance rejection that, compare to the modern control framework, are truly *active.* They are distinctly different from the standard methods in that the disturbance, mostly external, is estimated using an observer and canceled out, allowing control design to be reduced to one that is disturbance free. Thus the disturbance rejection problem is transformed to the disturbance observer design, a survey of which can be found in ([24]). To be sure, most of these disturbance observers, including the disturbance observer (DOB) ([2], [3], [23], [25]), the perturbation observer (POB) ([4], [15],[16], [20]), the unknown input observer (UIO) ([28], [11], [10]), can be traced to internal model control (IMC) ([27]), where the LTI model of the plant is explicitly used in the observer design. As for the NTV plants with both internal and external disturbances, the results are scarce. Two such solutions, model estimator (ME) ([29]) and time-delay control (TDC) ([12], [30]), are proposed where both the internal and external disturbances are estimated and rejected. The trade-off is they require the measurement and feedback of the derivatives of plant output up to the (n-1)th and (n+1)th order, respectively, making their practicality questionable.

This brings us to the focus of this paper, the active disturbance rejection control (ADRC) ([6],[13]), as the only comprehensive, systematic solution to disturbance estimation and rejection. It was first proposed by J. Han in 1995 ([13]) and is simplified and parameterized, and thus made practical, by Gao ([8]). ADRC has been successfully implemented in a wide range of applications including motion control ([5], [7], [9], [21], [22], [33]), jet engines control ([32]), MEMS Gyroscope control ([35]), and process control ([26]), etc.

Although ADRC has demonstrated the validity and advantage in many applications, its stability characteristics has not been fully understood. Bounded input and bounded output (BIBO) stability was suggested in [8]. Frequency domain stability analysis for linear plants is shown in [34]. The convergence and the bounds of the estimation and tracking errors are presented in [19]. Khalil in [17] applied output feedback variable structure control (VSC) along with a nonlinear high-gain observer to guarantee boundness of all variables of the closed-loop system, and output tracking of a given reference signal in the presence of modeling uncertainty and time-varying disturbance. One notice that the system in [17] is more general than the system in ADRC design, that implies the control algorithm in [17]'s designed needs more information than ADRC algorithm. Otherwise, the controller cannot be designed. The purpose of this paper is to show analytically how to use singular perturbation theory to obtain the sufficient condition of exponentially stability of the closed-loop system of ADRC, and thus, establish a lower bound for the observer bandwidth.

The paper is organized as follows. In Section 1, we review some of the background of ADRC and motivation behind this research. Section 2 presents a description of ADRC which will be analyzed in this study, which the error dynamic of the nth-order plant with ADRC is derived and formulated as a standard singular perturbed system. The exponentially stability condition is provided, and the lower bound for the observer bandwidth is determined to guarantee the exponential stability of the closed-loop system in Section 3. Some final remarks are given in Section 4.

2. Active Disturbance Rejection Control (ADRC)

In this section we give a brief review of ADRC and introduce the problem of stability analysis.. Most of the results are well known.

Consider a plant, described by a nth-order nonlinear differential equation with unknown dynamics and external disturbances described by

$$y^{(n)} = f(x, \dot{y}, \cdots, y^{(n-1)}, w, u) + bu \tag{2.1}$$

where u and y are the input and output of the plant, respectively. The external disturbance w is combined with unknown dynamics of the form $f(x, \dot{y}, ..., y^{(n-1)})$. For the sake of simplicity, we denote $f(\cdot) = f(x, \dot{y}, ..., y^{(n-1)}$, which is a general nonlinear, time-varying (NTV) dynamics and represents *the total disturbance.*

Instead of following the traditional design path of modeling to obtain an explicit mathematical expression of $f(\cdot)$, ADRC offers an alternative that greatly reduces the dependence on explicit modeling. The strategy is to actively estimate $f(\cdot)$ and then cancel it in real time. Thereby reducing the problem to the control of an integral plant. That is, if an estimate of $f(\cdot)$ is obtained as $\hat{f}(\cdot)$, then

$$u = -(\hat{f}(\cdot) + u_0)/b \tag{2.2}$$

reduces the plant in 2.1 to a cascade integral form:

$$y^{(n)} = f(\cdot) - \hat{f}(\cdot) + u_0 \approx u_0 \tag{2.3}$$

At this point, the original unknown NTV plant of (2.1) is transformed to a simple plant that is quite easy to control. This is the key idea and main benefit of ADRC. It only works, of course, if $f(\cdot)$ can be estimated effectively, which we will discuss next.

Before proceed to estimate $f(\cdot)$, we make the following two assumptions.

Assumption A: $f(\cdot)$ and its derivative $\eta(\cdot)$ are locally Lipchitz in their arguments and bounded within the domain of interest. In addition, the initial conditions are assumed such that $f(\cdot)|_{t=0} = 0$, and $\eta(\cdot)|_{t=0} = 0$.

Assumption B: That the desired output of (2.1) and its derivatives up to (n+2)th order are bounded.

Now we are ready to introduce the extended state observer.

The Extended State Observer (ESO)

To obtain $\hat{f}(\cdot)$, we extend the state vector to include $f(\cdot)$ as an additional state. That is, define

$$x = [x_1, x_2, ..., x_{n+1}]^T = [y, \dot{y}, ..., f(\cdot)]^T,$$

then the plant in (2.1) can be expressed as

$$\dot{x} = A_e X_e + B_e u + E\eta \tag{2.4}$$

where

$$A_e = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad B_e = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b \\ 0 \end{bmatrix}_{(n+1)\times 1}, \quad E = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n+1)\times 1},$$

and $\eta = f(\cdot)$. A state observer of (2.4), denoted as ESO, is constructed as a system of differential equation

$$\begin{aligned} \dot{z} &= A_e z + B_e u + l(y - \hat{y}) \\ \hat{y} &= Cz \end{aligned} \tag{2.5}$$

where $z = x = [z_1, z_2, ..., z_{n+1}]^T, l = [l_1, l_2, ..., l_{n+1}]^T, C = [1, 0, ..., 0]$.
By setting $\lambda(s) = |sI - (A_e - lC)| = (s + \omega_0)^{n+1}$, the observer gains are selected as $l = [\beta_1\omega_0, \beta_2{\omega_0}^2, \cdots, \beta_{n+1}{\omega_0}^{n+1}]^T$, where ω_0 is the bandwidth and the only tuning parameter ω_0 of the observer (see [6]), and $\beta_i, i = 1, 2, ..., n+1$, are chosen such that the roots of $s^{n+1} + \beta_1 s^n + \cdots + \beta_n s + \beta_{n+1} = 0$ are in the open left-half complex plane. In this case, if we put all the poles into the same pole location by let ω_0, we can easily obtain the coefficient of β_i for all i.

Define the estimation error vector of ESO as

$$\tilde{e} = x - z \tag{2.6}$$

Subtracting (2.5) from (2.4), the error dynamics of the ESO is as follows

$$\dot{\tilde{e}} = (A - lC)\tilde{e} + E\eta \tag{2.7}$$

For the purpose of parameterization and the stability analysis, we introduce the following change of coordinates,

$$(2.8)\qquad \begin{cases} \tilde{e}_1 = \omega_o \xi_1 \\ \vdots \\ \tilde{e}_n = \omega_o^n \xi_n \\ \tilde{e}_{n+1} = \omega_o^{n+1} \xi_{n+1} \end{cases}$$

Equation (2.8) can be written as

$$(2.9)\qquad \tilde{e} = \begin{bmatrix} \omega_o & 0 & 0 & \cdots & 0 \\ 0 & \omega_o^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \omega_o^i & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \omega_o^{n+1} \end{bmatrix} \xi = \Lambda \xi$$

where $\tilde{e} = [\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{n+1}]^T$, $\xi = [\xi_1, \xi_2, \ldots, \xi_{n+1}]^T$,

$$\Lambda = diag[\omega_{\text{o}}, \omega_o^2, \cdots, \omega_o^{n+1}],$$

$$\Lambda_i^{-1} = diag[\omega_{\text{o}}^{-1}, \omega_o^{-2}, \cdots, \omega_o^{-(n+1)}],$$

and ω_o is the design parameter. Substitute (2.9) into (2.7), we have

$$(2.10)\qquad \Lambda \dot{\xi} = (A_e - lC)\Lambda \xi + E\eta$$

Since matrix Λ is a diagonal matrix and invertible, equation (2.10) could be transformed as follows:

$$(2.11)\qquad \dot{\xi} = \Lambda^{-1}(A_z \Lambda \xi + \Lambda^{-1} E\eta),$$

where

$$(2.12)\qquad A_z = \begin{bmatrix} -\beta_1 & 1 & 0 & \cdots & 0 \\ -\beta_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n & 0 & 0 & \cdots & 1 \\ -\beta_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

(2.11) can be rewritten as

$$(2.13)\qquad \dot{\xi} = \omega_o A_z \xi + \omega_o^{-(n+1)} E\eta,$$

that is the error dynamic that to be used later.

Control Law

The control law is designed as

$$u = \frac{1}{b}(-z_{n+1} + u_0) \tag{2.14}$$

which reduces the plant (2.1) to approximate a n-order integral plant

$$y^{(n)} = (f - z_{n+1}) + u_0 \approx u_0 \tag{2.15}$$

which can be controlled using

$$u_0 = k_1(y_r - z_1) + k_2(\dot{y}_r - z_2) + \cdots + k_n(y_r^{(n)} - z_n) \tag{2.16}$$

For the tracking reason, we define the desired track state vector as

$$x_r = [y_r, \dot{y}_r, \ldots, {y_r}^{(n+1)}],$$

and the tracking error vector as follows

$$e = x - x_r = [e_1, e_2, \ldots, e_n]^T, \tag{2.17}$$

then the error dynamics of the plant (2.1) is as follows

$$\dot{e} = A_1 e + B_1 u + B_f f \tag{2.18}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times(n+1)}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}_{(n+1)\times 1}, \quad B_f = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{(n+1)\times 1},$$

From equation(2.6), it can be concluded that

$$z_{n+1} = f - \tilde{e}_{n+1} \tag{2.19}$$

Substitute (2.16) and (2.19) into (2.14), and we obtain the control input

$$u = \frac{1}{b}[-Ke + K_f \tilde{e} - f] \tag{2.20}$$

where $K = [k_1, k_2, \ldots, k_n]^T$, which is designed to make $A_f = A_1 - B_f K$ a Hurwitz matrix, and $K_f = [K, 1]^T$.

Substitute controller defined by (2.20) into error dynamics (2.13), the error dynamics of the closed-loop system is as follows:

$$\dot{e} = A_f e + B_f K_f \Lambda \xi \tag{2.21}$$

Combine the observer error dynamics by (2.13), we obtain and the tracking error dynamics by (2.21), we can get the error dynamics of the closed-loop system as follows:

$$\begin{cases} \dot{e} = A_f e + B_f K_f \Lambda \xi, \\ \dot{\xi} = \omega_o A_z \xi + \omega_o^{-(n+1)} E\eta. \end{cases} \tag{2.22}$$

As the close-loop system dynamics, (2.22) is thereby served as the starting point for the next step stability analysis. We are particularly interested in knowing when under what condition (2.22) is asymptotically stable, and what is the relationship between ω_0 and the performance (error) of the control system.

3. STABILITY ANALYSIS

In this section we study stability characteristics of ADRC. In particular, we wish to determine the stability condition of the closed-loop error dynamics described by (2.22). This is guided by the insight that the observer dynamics, the second equation in (2.22), is usually much faster than that of the state feedback. The task of analysis is perhaps made easier if we can separate the fast dynamics from the slow one, and this is a problem dealt with in singular perturbation theory. Specifically, a nonlinear system is said to be singularly perturbed if it can be expressed in the following form:

$$\begin{cases} \dot{x} = f(t, x, z, \varepsilon), \quad x(t) = x_0, \quad x \in R^n \\ \varepsilon \dot{z} = g(t, x, z, \varepsilon), \quad z(t) = x_0, \quad z \in R^m \end{cases} \tag{3.1}$$

where ε represents a small parameters to be neglected. The functions f and g are assumed to be sufficiently smooth, that is, they are continuously differentiable functions of all variables t, x, z, and ε. Throughout the paper, we assume that $||x||$ is defined as the Euclidean norm of x in l^2 space. Note that (2.22) can be easily formulated in the form of (3.1) if ε is chosen as $\varepsilon = 1/\omega_0$, which results

$$\begin{cases} \dot{e} = A_f e + B_f K_f \Lambda \xi \\ \varepsilon \dot{\xi} = A_z \xi + \varepsilon^{n+2} E\eta \end{cases} \tag{3.2}$$

Clearly, the singular perturbation theory developed for dynamic systems of the form of (3.1) can now be applied to analyze stability of (3.2). We first state the following three theorems (Theorem 3.1, Theorem 3.2 and Theorem 3.3) from [18] and [19]), which proved to be especially useful to our stability analysis:

Theorem 3.1. *Consider the singularly perturbed system*

$$\begin{cases} \dot{x} = f(t,x,z,\varepsilon) \\ \varepsilon\dot{z} = g(t,x,z,\varepsilon) \end{cases} \tag{3.3}$$

Assume that the following assumptions are satisfied for all $(t,x,\varepsilon) \in [0,\infty) \times B_r \times [0,\varepsilon_0]$, *where* $B_r = \{x \in R^n | \ ||x|| \leq r\}$ *with* $r > 0$.

- *1.* $f(t,0,0,\varepsilon) = 0$ *and* $g(t,0,0,\varepsilon) = 0$.
- *2. The equation* $g(t,x,z,0) = 0$ *has an isolated root* $z = h(x,t)$ *such that* $h(t,0) = 0$.
- *3. The functions* f, g, h *and their derivatives are bounded up to the second order for* $z - h(x,t) \in B_\rho$, *where* $B_\rho = \{s \in R^m | \ ||s|| \leq \rho \leq r\}$ *with* $0 < \rho \leq r$.
- *4. The origin of the reduced system* $\dot{x} = f(t,x,h(x,t),0)$ *is exponentially stable.*
- *5. The origin of the boundary-layer system* $\frac{dy}{d\tau} = g(\tau,x,y+h(x,\tau),0)$ *is uniformly and exponentially stable in* (x,τ), *where* τ *is a scaling variable of* t *when* $\varepsilon > 0$ *is small.*

Then, there exists $\varepsilon^* > 0$ *such that for all* $\varepsilon < \varepsilon^*$, *the origin of (3.3) is exponentially stable.*

Theorem 3.2. *Let* $x = 0$ *be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t,x) \tag{3.4}$$

where $f : [0,\infty)\times D \to R^n$ *is continuous differentiable,* $D = \{x \in R^n | \, \|x\| < r\}$ *and the Jacobian matrix* $[\partial f/\partial x]$ *is bounded uniformly on* D *in* t. *Let* k, λ *and* r_0 *be positive constants with* $r_0 < r/k$, *and define* $D_0 = \{x \in R^n | \, \|x\| < r_0\}$. *Assume that the trajectories of the system satisfy*

$$\|x(t)\| \leq k \, \|x(t_0)\| \, e^{-\lambda(t-t_0)}, \forall x(t_0) \in D_0, \forall t \geq t_0 \geq 0. \tag{3.5}$$

Then there is a function $V : [0,\infty) \times D_0 \to R$ *that satisfies the inequalities*

$$c_1 \|x\|^2 \leq V(t,x) \leq c_2 \|x\|^2 , \tag{3.6}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -c_3 \|x\|^2 , \tag{3.7}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| , \tag{3.8}$$

for some positive constants c_1, c_2, c_3 *and* c_4. *Moreover, if* $r = \infty$ *and the origin is globally exponentially stable, then* $V(t,x)$ *satisfies the aforementioned* *inequalities on R^n. Furthermore, if the system is autonomous, $V(t,x)$ can be chosen independent of t.*

Now consider the slow system

$$\dot{x} = f(x,u), \tag{3.9}$$

where $x \in R^n$ and $u \in \Gamma \subset R^m$ for all $t \geq 0$. Suppose $f(x,u)$ is locally Lipschitz on $R^n \times \Gamma$ for every $u \in \Gamma$, the equation (3.9) has a continuously differentiable isolated root.

To analyze the stability properties of the frozen equilibrium point $x = h(\alpha)$, we shift it to the origin via the change of variables $z = x - h(\alpha)$ to obtain the equation

$$\dot{z} = f(x + h(\alpha), \alpha) \stackrel{def}{=} g(z,\alpha). \tag{3.10}$$

Then we have the following theorem.

Theorem 3.3. *Consider the system (3.10), suppose $g(z,\alpha)$ is continuous differentiable and the Jacobian matrices $[\partial g/\partial z]$ and $[\partial g/\partial z]$ satisfy*

$$\left\| \frac{\partial g}{\partial z}(z,\alpha) \right\| \leq L_1, \left\| \frac{\partial g}{\partial \alpha}(z,\alpha) \right\| \leq L_2 \|z\| \tag{3.11}$$

for all $(z,\alpha) \in D \times \Gamma$ where $D = \{z \in R^n | \, \|z\| < r\}$. Let k, γ, and r_0 be positive constants with $r_0 < r/k$, and define $D_0 = \{z \in R^n | \, \|z\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|z(t)\| \leq k \|z(0)\| e^{-\gamma t}, \forall z(0) \in D_0, \alpha \in \Gamma, \forall t \geq 0, \tag{3.12}$$

then there is a function $W : D_0 \times \Gamma \to R$ that satisfies (3.13) through (3.16). Moreover, if all the assumptions hold globally in z, then $W(z,\alpha)$ is defined and satisfies (3.13) through (3.16) on $R^n \times \Gamma$

$$b_1 \|z\|^2 \leq W(z,\alpha) \leq b_2 \|z\|^2, \tag{3.13}$$

$$\frac{\partial W}{\partial z} g(z,\alpha) \leq -b_3 \|z\|^2, \tag{3.14}$$

$$\left\| \frac{\partial W}{\partial z} \right\| \leq b_4 \|z\|, \tag{3.15}$$

$$\left\| \frac{\partial W}{\partial \alpha} \right\| \leq b_5 \|z\|, \tag{3.16}$$

for all $z \in D = \{z \in R^n | \, \|z\| < r\}$ and $\alpha \in \Gamma$, where $b_i, i = 1, \ldots, 5$ are positive constants independent of α.

Before proceeding to derive our two main theorems in stability analysis, we further assume that system (2.1) satisfies Assumption A and Assumption B of Section 2. Recall

Assumption A: It is assumed that $f(\cdot)$ and its derivative $\eta(\cdot)$ are locally Lipschitz in their arguments and bounded within the domain of interest. Furthermore, the initial conditions $f(\cdot)|_{t=0}$=0, and $\eta(\cdot)|_{t=0} = 0$.

Assumption B: It is assumed that the desired output of (2.1) and its derivatives up to $(n+2)th$ order are bounded.

Applying Theorem 3.1 to (3.2), we obtain our first main result in stability analysis.

Theorem 3.4. *Consider the ADRC error dynamics in (3.2). Let Assumption A and Assumption B hold for (2.1), then there exists an $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$, the origin of (3.2) is exponentially stable.*

Proof: In order to apply Theorem 3.1, we need to show is that system (3.2) satisfies all five assumptions of Theorem 3.1. Comparing (3.2) and (3.3), it is clearly that

(3.17) $$f = A_f e + B_f K_f \Lambda \xi$$

(3.18) $$g = A_z \xi + \varepsilon^{n+2} E \eta$$

By the definitions of f, g, and Assumption A, $\dot{\eta()} = 0$. One can easily see that Assumptions 1 is satisfied.

For Assumption 2, we need to separate the slow and fast model from the original system defined by (3.2) and follow the procedures described in [18]. To obtain the quasi-steady-state model, let $\varepsilon = 0$, and solve the algebraic equation:

(3.19) $$A_z \xi + \varepsilon^{(n+2)} E \eta = 0.$$

The solution of (3.19) has the form

(3.20) $$\bar{\xi} = \phi(\bar{e}, t) = 0$$

Obviously, $\bar{\xi}$ is an isolated root for (3.20), and Assumption 2 is therefore satisfied.

To check Assumption 3, we need to show that function f, g, ϕ and their partial derivatives are bounded. Since e and ξ vanish at the origin for all $\varepsilon \in [0, \varepsilon_0]$

, they are Lipschitz in ε linearly in the state (e, ξ) ([18]). By Assumption A and Assumption B: both $\eta(\cdot)$ and $\dot{\eta}(\cdot)$ are bounded, we have

$$\|B_f K_f \Lambda \xi\| \leqslant L_1 \|\xi\| \tag{3.21}$$

$$\|\eta\| \leqslant L_2(\|e\| + \|\xi\|) \tag{3.22}$$

$$\|\dot{\eta}\| \leqslant L_3(\|\dot{e}\| + \left\|\dot{\xi}\right\|) \tag{3.23}$$

where L_1, L_2 and L_3 are positive constants. Hence, we can show that f, g, ϕ and their partial derivatives are bounded:

$$f = A_f e + B_f K_f \Lambda \xi \leqslant A_f \|e\| + L_1 \|\xi\| \tag{3.24}$$

$$\dot{f} = A_f \dot{e} + B_f K_f \Lambda \dot{\xi} \leqslant A_f \|\dot{e}\| + L_1 \left\|\dot{\xi}\right\| \tag{3.25}$$

$$g = A_z \xi + \varepsilon^{n+2} E \eta \leqslant A_z \|\xi\| + \varepsilon^{n+2} E L_2(\|e\| + \|\xi\|) \tag{3.26}$$

$$\dot{g} = A_z \dot{\xi} + \varepsilon^{n+2} E \dot{\eta} \leqslant A_z \left\|\dot{\xi}\right\| + \varepsilon^{n+2} E L_3(\|\dot{e}\| + \left\|\dot{\xi}\right\|) \tag{3.27}$$

$$\phi(\bar{e}, t) = 0. \tag{3.28}$$

Therefore, we conclude that Assumption 3 is satisfied.

Substitute (3.20) into the first equation of (3.2), we obtain the quasi-steady-state model as follows:

$$\dot{e} = A_f e \tag{3.29}$$

Since A_f is a Hurwitz matrix, it is obvious that Assumption 4 holds.

The boundary layer system, which is the fast dynamics, is obtained by introducing a time scale of

$$\tau = t/\varepsilon \tag{3.30}$$

As $\varepsilon \to 0$, we have

$$\frac{d\xi}{d\tau} = A_z(\tau)\xi(\tau). \tag{3.31}$$

This is the fast dynamics of (3.2). Since A_z is a Hurwitz matrix, it is obvious that Assumption 5 holds.

Note that all the assumptions are satisfied. By Theorem 3.1, the origin of (3.2) is exponentially stable. We finish the proof.

Remarks

1. Theorem 4 indicates that there exists an upper bound of ε for (3.2) to be exponentially stable, which means that observer bandwidth ω_o must have a lower bound.

2. The advantage of the proposed approach is that the singular perturbation analysis divides the original problem into two systems: the slow subsystem or quasi-steady state system and the fast subsystem or boundary-layer system. The separated system can then be studied independently.

3. Theorem 3.1 shows proved that there exists a certain ε^* that can guarantee the origin of (3.2) is exponentially stable. Since (3.2) is a linear system, our next goal is to obtain the upper bound of ε explicitly.

In the spirit of Theorem 3.2 and Theorem 3.3, we are ready to introduce our next main result to obtain the upper bounds of ε in Theorem 3.5.

Theorem 3.5. *Consider singular perturbed system (3.2). Assume that Assumption A and Assumption B hold for (2.1), then there exist an upper bond of* ε^**, such that*

$$\varepsilon^* \leq \min\left(\sqrt[n+1]{\left[2(1-d)c_3 - c_4L_1(1-d)\right]/db_4L_2}, \frac{db_3}{c_4L_1(1-d)}, \sqrt[n+1]{\frac{1}{3}\frac{c_4L_1(1-d)}{db_4L_2}} \right)$$

where $b_i (i = 1, \ldots, 4)$*,* $c_i (i = 1, \ldots, 4)$*,* L_1*, and* L_2 *are nonnegative constants,* $0 < d < 1$*. Then for all* $\varepsilon \leq \varepsilon^*$*, the origin of (3.2) is exponentially stable.*

Proof: By Theorem 3.3, there is a Lyapunov function $V(e)$ for the reduced system that satisfies

$$c_1 \|e\|^2 \leq V(e) \leq c_2 \|e\|^2, \tag{3.32}$$

$$\frac{\partial V}{\partial x} A_f e \leq -c_3 \|e\|^2, \tag{3.33}$$

$$\left\| \frac{\partial V}{\partial e} \right\| \leq c_4 \|e\|, \tag{3.34}$$

for some positive constants $c_i, i = 1, \ldots, 4$ and for $e \in B_{r0}$ with $r_0 \leq r$.

From Theorem 3.4, there is a Lyapunov function $W(\xi)$ for the boundary layer system that satisfies

$$b_1 \|\xi\|^2 \leq W(\xi) \leq b_2 \|\xi\|^2, \tag{3.35}$$

$$\frac{\partial W}{\partial \xi} A_z \xi \leq -b_3 \|\xi\|^2, \tag{3.36}$$

$$\left\| \frac{\partial W}{\partial \xi} \right\| \leq b_4 \left\| \xi \right\|, \tag{3.37}$$

for some positive constants $b_i (i = 1, \ldots, 4)$ and for $\xi \in B_{\rho_0}$ with $\rho_0 \leq \rho$.

Since e and ξ vanish at the origin for all $\varepsilon \in [0, \varepsilon_0]$, they are Lipschitz in ε linearly in the state (e, ξ). In particular,

$$\left\| B_f K_f \Lambda \xi \right\| \leq L_1 \left\| \xi \right\| \tag{3.38}$$

$$\left\| E\eta \right\| \leq L_2 \left(\left\| e \right\| + \left\| \xi \right\| \right) \tag{3.39}$$

where L_1 and L_2 are two positive constants.

We set

$$V_{cl}(e, \xi) = (1-d)V(e) + dW(\xi) \tag{3.40}$$

as a Lyapunov function candidate for system (3.2), where d is a weighting variable, $0 < d < 1$. By the properties of functions and using the estimates from (3.32) to (3.39), one can verify that the derivative of (3.40) along the trajectories of (3.2) satisfies the following inequalities:

$$\begin{aligned}
\dot{V}_{cl} &= (1-d)\frac{\partial V}{\partial e}(A_f e + B_f K_f \Lambda \xi) + d\frac{\partial W}{\partial \xi}(\frac{1}{\varepsilon}A_z \xi + E\eta) \\
&= (1-d)\frac{\partial V}{\partial e}A_f e + (1-d)\frac{\partial V}{\partial e}B_f K_f \Lambda \xi + d\frac{\partial W}{\partial \xi}\frac{1}{\varepsilon}A_z \xi + d\frac{\partial W}{\partial \xi}E\eta \\
&\leq -(1-d)c_3 \left\| e \right\|^2 + (1-d)c_4 \left\| e \right\| L_1 \left\| \xi \right\| - \frac{d}{\varepsilon}b_3 \left\| \xi \right\|^2 \\
&\quad + db_4 \left\| \xi \right\| \left[L_2 \left(\left\| e \right\| + \left\| \xi \right\| \right) \right] \\
&\leq -(1-d)c_3 \left\| e \right\|^2 - \frac{d}{\varepsilon}b_3 \left\| \xi \right\|^2 + db_4 \varepsilon^{n+1} L_2 \left\| \xi \right\|^2 \\
&\quad + \left(c_4 L_1 (1-d) + db_4 \varepsilon^{n+1} L_2 \right) \left\| e \right\| \left\| \xi \right\| \\
&\leq -(1-d)c_3 \left\| e \right\|^2 + \left[db_4 \varepsilon^{n+1} L_2 - \frac{d}{\varepsilon}b_3 \right] \left\| \xi \right\|^2 \\
&\quad + \left(c_4 L_1 (1-d) + db_4 \varepsilon^{n+1} L_2 \right) \left(\frac{\left\| e \right\|^2 + \left\| \xi \right\|^2}{2} \right) \\
&\leq \left[-(1-d)c_3 + \frac{1}{2}c_4 L_1 (1-d) + \frac{1}{2}db_4 \varepsilon^{n+1} L_2 \right] \left\| e \right\|^2 \\
&\quad + \left[\frac{3}{2}db_4 \varepsilon^{n+1} L_2 - \frac{d}{\varepsilon}b_3 + \frac{1}{2}c_4 L_1 (1-d) \right] \left\| \xi \right\|^2 \\
&\leq -\mu_1 \left\| e \right\|^2 - \mu_2 \left\| \xi \right\|^2 \qquad (3.41)
\end{aligned}$$

where $\begin{cases} \mu_1 = (1-d)c_3 - \frac{1}{2}c_4 L_1(1-d) - \frac{1}{2}db_4\varepsilon^{n+1}L_2 \\ \mu_2 = \left[\frac{d}{\varepsilon}b_3 - \frac{1}{2}c_4 L_1(1-d) - \frac{3}{2}db_4\varepsilon^{n+1}L_2\right] \end{cases}$

In order to obtain the desired $v_{cl} \leq 0$, we need to make both μ_1 and μ_2 be positive. From (3.41), we first let $\mu_1 \geq 0$, then

$$\begin{aligned} &\mu_1 = (1-d)c_3 - \frac{1}{2}c_4 L_1(1-d) - \frac{1}{2}db_4\varepsilon^{n+1}L_2 \geq 0 \\ &\Rightarrow \varepsilon_1^* \leq \sqrt[n+1]{\left[2(1-d)c_3 - c_4 L_1(1-d)\right]/db_4 L_2}. \end{aligned} \tag{3.42}$$

Nexy let $\mu_2 \geq 0$, we have

$$\begin{aligned} &\mu_2 = \left[\frac{d}{\varepsilon}b_3 - \frac{1}{2}c_4 L_1(1-d) - \frac{3}{2}db_4\varepsilon^{n+1}L_2\right] \geq 0 \\ &\Rightarrow \begin{cases} \frac{d}{\varepsilon}b_3 - \frac{1}{2}c_4 L_1(1-d) \geq \frac{1}{2}c_4 L_1(1-d) \\ \frac{1}{2}c_4 L_1(1-d) \geq \frac{3}{2}db_4\varepsilon^{n+1}L_2 \end{cases} \\ &\Rightarrow \varepsilon_2^* \leq \min\left(\frac{db_3}{c_4 L_1(1-d)}, \sqrt[n+1]{\frac{1}{3}\frac{c_4 L_1(1-d)}{db_4 L_2}}\right) \end{aligned} \tag{3.43}$$

Based on the selection of ε_1^* and ε_2^*, it is guaranteed that

$$\dot{V}_{cl} \leq -\min(\mu_1, \mu_2)\left[\|e\|^2 + \|\xi\|^2\right] \tag{3.44}$$

which completes the proof.

Remarks

(1). Theorem 3.5 is an extension of Theorem 3.2, in which equation (3.42)-(3.44) determine the upper bound of ε. Since $\varepsilon = 1/\omega_o$, it means that the lower bound of observer bandwidth ω_o can be obtained based on Theorem 3.5.

(2). The ESO in the fast time scale τ is faster than the dynamics of the plant and the controller, we are able to make the estimated state converge to the real state faster. This explains why ESO can actively reject the disturbance, since the extended state can estimate the unknown dynamics very well.

The above results show that for the closed-loop system, when controlled by ESO and ADRC control law, presented in (2.22) achieves exponentially asymptotic convergence of the tracking errors.

4. CONCLUSION

This paper we presents a singular perturbation approach to analyze the stability characteristics of the ADRC control system for nonlinear time-variant plant. The closed-loop system is reformulated to allow the application of singular perturbation method, which enables the decomposition of the original system into a relatively slow subsystem and a relatively fast subsystem. Based on the decomposed subsystems, the composite Lyapunov function method is used to show that the closed-loop system, achieves exponentially stable under certain conditions. In the framework of singular perturbation, the observer error and the tracking error of the system are exponentially stable $\varepsilon = 0$. Since it is practically impossible, we establish the existence of the lower bound for the observer bandwidth that guarantees the exponential stability of the closed-loop system.

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Received: July, 2008