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Optimal fast tracking observer bandwidth of the linear extended state observer

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In current industrial control applications, the proportional + integral + derivative (PID) control is still used as the leading tool, but constructing controller requires precise mathematical model of plant, and tuning the parameters of controllers is not simple to implement. Motivated by the gap between theory and practice in control problems, linear active disturbance rejection control (LADRC) addresses a set of control problems in the absence of precise mathematical models. LADRC has two parameters to be tuned, namely, a closed-loop bandwidth and observer bandwidth. The performance of LADRC depends on the quick convergence of a unique state observer, known as the extended state observer, proposed by Jinqing Han (1994). Only one parameter, observer bandwidth, significantly affects the tracking speed of extended state observer. This paper studies numerically the optimal fast tracking observer bandwidth and the absolute tracking error estimation for a class of non-linear and uncertain motion control problems by finite difference method.

1. Introduction

The growth of digital control is a contemporary nature. In a digital implementation, the controller collects samples through the sensor, compares it to the reference and computes the corresponding input to the plant based on the control law. The state feedback control laws require access to all state variables. For those states that are not directly measured, their estimates, obtained from a state observer, are used. The problem of observer design is directed to finding a mechanism to estimate the unmeasured states from the available output measurements. Consequently, observer design has become a key factor in control design. Luenberger (1964, 1966, 1971) introduced the state observer for linear systems, known as the Luenberger Observer. For a non-linear system, several methods have been proposed. Misawa and Hedrick (1989) surveyed some of these methods. The performance of these observers and the resulting control system largely depend on the accuracy of the mathematical model of the plant, which poses a practical concern. To address this issue, an ingenious observer system, known as extended state observer (ESO), was proposed by Han (1989, 1994, 1995), where the states as well as the uncertainties in the plant are estimated. This allows the controller to actively compensate for the uncertainties, and it led to the active disturbance rejection control (ADRC) (Gao *et al*. 2001a, b, Wang *et al*. 2003, Gao *et al*. 2004). The strong robustness of the non-linear ESO (NLESO) has attracted the attention of authors in recent year (Huang *et al*. 2001, 2002a, b). The outstanding performance of NLESO among the other observers was verified by the experiments. However, the number of parameters to be tuned, and the computational burden are still the problem to resolve in NLESO. By using a linear feedback instead of a non-linear one, Gao (2003, 2004) and Gao *et al*. (2001a) proposed the linear active disturbance rejection control (LADRC) with linear extended state observer (LESO) for the single-input single-output (SISO) non-linear uncertain system. LADRC is easy to use and to tune because it has only two tuning parameters, namely,

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the closed-loop bandwidth and the observer bandwidth. However it requires that LESO converges quickly to the actual signal. Only observer bandwidth significantly affects the tracking speed of extended state observer.

In this paper, tracking effectiveness of observer bandwidth and optimal fast tracking observer bandwidth are studied under a set of practical motion control problems with digital implementation.

2. Preliminary of LADRC

2.1 *Practical motion control problem*

The modern control deals directly with systems in ordinary differential equation. In a typical application using a motor as the power source, the plant equation of motion can be described as

$$\ddot{y}(t) = f(t, y(t), \dot{y}(t), w(t)) + b(t)u(t), \tag{1}$$

where for all $t \geq 0$ the signal $y(t)$ is the position output, $u(t)$ is the voltage to the power amplifier, $b(t)$ is a time-varying coefficient, and $w(t)$ represents the unknown external disturbance such as vibrations and torque disturbances. The friction, the effect of inertia and various other non-linearities in a motion system are all represented by the uncertain function $f(\cdot)$. So, $f(\cdot)$ is a time-varying function in practical sense. The desired trajectory of the position is known, and there are many constraints in motion control design. An example is the limitation of motor torque, and if the digital controller is used, which is implemented by a computer, the sampling frequency is delimited by its electromechanical system. The term $b(t)$ in (1), which is related to both the inertia of the object to be moved and the motor torque constant, usually changes slowly, so that it is continuous and differentiable.

2.2 *Active disturbance rejection control*

2.2.1 Han's active disturbance rejection control (ADRC) for (1) (Han 1998, Gao *et al.* 2001a, b)**:**. Consider the plant dynamics in (1) normalized at $b(t) = 1$ and let $f(\cdot)$ absorb the discrepancy $(b(t) - 1)u(t)$. Let $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, and

$$a(t) = f(t, x_1(t), x_2(t), u(t), w(t)). \tag{2}$$

Then, a state space description of (1) is

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= a(t) + u(t) \\ y(t) &= x_1(t). \end{aligned}\right\} \tag{3}$$

Let $x_3(t) = a(t)$ be an additional state variable in (3) and let $h(t) = \dot{a}(t)$. The problem of extended state observer design is to reconstruct the state x_2 and the extended state x_3 via $u(t)$ and $y(t)$. The state space description (3) can be rewritten as

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) + u(t) \\ \dot{x}_3(t) &= h(t) \\ y(t) &= x_1(t). \end{aligned}\right\} \tag{4}$$

The key here is that the state augmentation in (4) allows $a(t)$ to be estimated as a state x_3. The control problem formulation takes a sharp turn here: instead of trying to find $f(t, y(t), \dot{y}(t), u(t), w(t))$ in system identification, estimates it and compensates for it in real time! This is the basis of ADRC.

2.3 *LADRC tuning method*

Han's ADRC method (Han 1998, Su *et al.* 2002) shows a huge success in dealing with some of the most challenging industrial control problems but it has many parameters to be tuned. So, one remaining task to field engineer is how to simplify the tuning process. Gao (Gao 2003, Gao *et al.* 2001a) proposed a tuning method using linear gains in ADRC which reduces tuning parameters to only two, namely, the closed-loop bandwidth ω_c and the observer bandwidth ω_o.

Gao's tuning method can be described as follows for a 3-dimensional problem.

Write (4) as

$$\left.\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{E}h(t), \\ y(t) &= \mathbf{C}x(t), \end{aligned}\right\} \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and $\mathbf{E} = [0 \quad 0 \quad 1]^T$. Note that $[\ ]^T$ denotes transpose. The state space observer, denoted as the linear extended state observer (LESO), of (5) is constructed as

$$\left.\begin{aligned} \dot{z}(t) &= \mathbf{A}z(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \hat{y}(t)), \\ \hat{y} &= \mathbf{C}z(t), \end{aligned}\right\} \tag{6}$$

and $\mathbf{L}$ is the observer gain vector, which can be obtained by using any known method such as the pole placement technique,

$$L = [\beta_1 \quad \beta_2 \quad \beta_3]^T. \tag{7}$$

With the state observer properly designed and the controller is given by

$$\left.\begin{aligned} u(t) &= -z_3(t) + u_0(t) \\ u_0(t) &= k_p(v(t) - z_1(t)) - k_d z_2(t), \end{aligned}\right\} \tag{8}$$

where k_d is the gain of the derivative controller, k_p is the gain of the proportional controller, and $v(t)$ is the desired trajectory of the position. With the PD gains chosen as

$$k_d = 2\xi\omega_c \quad \text{and} \quad k_p = \omega_c^2, \tag{9}$$

where ω_c and ξ are the desired closed-loop bandwidth and damping ratio. The closed-loop transfer function $G_{\text{cl}}(s)$ in Laplace transform is approximately a standard second order transfer function

$$G_{\text{cl}}(s) = \frac{k_p}{s^2 + k_d s + k_p} = \frac{\omega_c^2}{s^2 + 2\xi\omega_c s + \omega_c^2}. \tag{10}$$

The ratio ξ can be conveniently set to unity to avoid any overshoot in the response and to allow the closed loop bandwidth, ω_c, the only tuning parameter to be adjusted in implementation. Furthermore, the observer gains can be obtained using any known method such as the pole placement technique. If the observer gains are chosen as

$$L = [\,3\omega_o \quad 3\omega_o^2 \quad \omega_o^3\,]^T, \tag{11}$$

the characteristic polynomial of the observer is

$$\lambda_o(s) = s^3 + \beta_1 s^2 + \beta_2 s + \beta_3 = (s + \omega_o)^3 \tag{12}$$

which means that the observer bandwidth, ω_o, is the only tuning parameter in the observer. Therefore, the number of tuning parameters in ADRC reduces to two.

3. Formulation of fast tracking problem

3.1 *Formulation of minimization problem*

As shown in §2.3, LADRC has the simple structure with only two tuning parameters. However it showed excellent performances and it was proved through the experiments (Gao 2003). Uncertainties of the function $f(\cdot)$ are eliminated in real time, and are compensated by the control signal if LESO is properly designed. However, LESO in LADRC requires that the estimate signal $z(t)$ quickly traces the actual signal $x(t)$. For this reason, it is of interest to know under what conditions $z(t)$ tracks the actual signal $x(t)$ and, if the conditions exist, which condition attains the best solution making a faster tracking basis. Thus the principle aim of this paper can be represented in a minimization problem, namely, fast tracking problem as follows.

Problem 1: Given any positive number ε, minimize t_0 such that $\|x(t) - z(t)\|_2 < \varepsilon$, for all $t > t_0$ subject to: state-space representation (5) and (6), two tuning parameters ω_c and ω_o, controller (8) and known desired trajectory of the position $v(t)$, unknown function $f(t, x_1(t), x_2(t), u(t), w(t))$.

The challenging task is, if there exists a feasible solution for the fast tracking problem, how to express it in terms of ω_c and ω_o. In §4, a finite difference method is used to analyse the feasible solution of this fast tracking problem.

3.2 *Assumptions of uncertain function*

The precise mathematical model of the function $f(\cdot)$ is usually unavailable in practice. The performance of plant is subject to the characteristics of control system. However, in most motion control applications, we may assume that the function $f(\cdot)$ has the following properties:

Assumption 1: $|f(t, y(t), \dot{y}(t), u(t), w(t))| \le k_1$ *for a known constant* k_1 *and all* $t \ge 0$.

Assumption 2: $f(\cdot)$ *is piecewise continuous on an interval* $0 \le t \le T$ *for any* $T > 0$ *where the interval is partitioned by a finite number of points* $0 = t_0 < t_1 < \cdots < t_n = T$.

Assumption 3: $f(\cdot)$ *is continuous and differentiable on each subinterval* $t_{i-1} < t < t_i$ *such that* $|(d/\mathrm{d}t) f(t, y(t), \dot{y}(t), w(t))| \le k_2$ *for a known constant* k_2.

Practically speaking, the ranges of $f(t, y(t), \dot{y}(t), u(t), w(t))$ and $b(t)$ are usually known but their analytical expressions are hard to come by.

4. Numerical approach to the estimation error

4.1 *Discretization of estimation error*

To approximate the error between state estimation (z) and the actual state (x), let the estimation error $e(t) = x(t) - z(t)$ where $e(t)$, $x(t)$ and $z(t)$ are 3×1 column vectors. Then, from (5) and (6), the tracking error dynamics becomes

$$\dot{e}(t) = (\mathbf{A} - \mathbf{LC})e(t) + \mathbf{E}h(t). \tag{13}$$

The closed-loop bandwidth will not be used to find tracking conditions for observer because the closed-loop bandwidth is used only in the control term $u(t)$, which does not appear on the tracking error dynamics. Thus we need only the observer bandwidth to approximate the tracking error. Therefore, Problem 1 becomes Problem 2.

Problem 2: Given any positive number ε, minimize t_0 such that $\|e(t)\|_2 < \varepsilon$, for all $t > t_0$ subject to: system representation (13), one tuning parameter ω_o, known desired trajectory of the position $v(t)$, and unknown function $f(\cdot)$ satisfying Assumptions 1–3.

Since we consider a digital implementation on the motion control with LESO, the system (13) can be discretized to find feasible solutions satisfying the fast tracking problem. Assume the time step size for z is equal to the one for x so that we can define $\Delta_t > 0$ as the time step size for e. By the numerical approach using Euler's forward method, the discretized form of (13) is

$$\begin{aligned} e[k] &= (\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))e[k-1] + \mathbf{E}(a[k] - a[k-1]) \\ &= (\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^k e[0] + \sum_{i=1}^{k} (\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^{k-i} \\ &\quad \times \mathbf{E}(a[i] - a[i-1]), \end{aligned} \tag{14}$$

where

$$(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC})) = \begin{bmatrix} 1 - 3\Delta_t\omega_o & \Delta_t & 0 \\ -3\Delta_t\omega_o^2 & 1 & \Delta_t \\ -\Delta_t\omega_o^3 & 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad h[k-1] = \frac{a[k] - a[k-1]}{\Delta_t}$$

and $k = \lfloor t/\Delta_t \rfloor > 0$. As long as the right hand side in (14) approaches to zero, it is clear that $e[k]$ converges to limit zero. Thus, to have feasible solutions to the fast tracking problem, we need the following assumption:

Assumption 4: *The initial tracking error* $e[0]$ *is bounded.*

In particular, the spectral radius of $(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))$, defined as $\rho(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC})) = \max_{\lambda \in \sigma(\mathbf{I}+\Delta_t(\mathbf{A}-\mathbf{LC}))} |\lambda|$, plays a central role of fast tracking estimation, where $\sigma(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))$ is the set of distinct eigenvalues.

Theorem 1 (convergence to zero)**:** *For matrix* $M \in C^{n\times n}$, $\lim_{k\to\infty} M^k = 0$ *if and only if* $\rho(M) < 1$.

Thus, the spectral radius of $(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))$ must be less than 1 such that $(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^k e[0]$ exponentially decays independent of $e[0]$ as $k \to \infty$. The spectral radius of $(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))$ is

$$\rho(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC})) = |1 - \Delta_t\omega_o|. \tag{15}$$

It should satisfy $|1 - \Delta_t\omega_o| < 1$. Therefore, the first constraint for the convergent condition requires

$$0 < \omega_o < (2/\Delta_t). \tag{16}$$

Assumption 5: *If observer bandwidth* ω_o *is selected within the constraint* (16), *there exists* $k > n_0$ *where* n_0 *is a sufficiently large natural number such that* $\|(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^k C\|_2 < (\varepsilon/3)$ *for any positive number* ε *and a fixed constant* C.

Using Assumptions 4 and 5 with the constraint (16), the first term of (14) can be upper bounded as

$$\left\|(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^k e[0]\right\|_2 < (\varepsilon/3). \tag{17}$$

It shows that the effect of initial tracking error $e[0]$ decays as $k > n_0$. For the sake of simplicity, introduce 3×1 vector $\mathbf{G}^i$ with its row components, G_1^i, G_2^i, and G_3^i such that

$$(\mathbf{I} + \Delta_t(\mathbf{A} - \mathbf{LC}))^i \mathbf{E} = \mathbf{G}^i = \begin{bmatrix} G_1^i & G_2^i & G_3^i \end{bmatrix}^T. \tag{18}$$

From the upper bound (17), we can induce that the behaviour of the tracking error $e[k]$ is dominated by the second term in (14) as $k > n_0$.

$$\|e[k]\|_2 < \left\| \sum_{i=1}^{k} \mathbf{G}^{k-i}(a[i] - a[i-1]) \right\|_2 + (\varepsilon/3). \tag{19}$$

Applying (18) to Assumption 5, we can get

$$\|\mathbf{G}^k C\|_2 < (\varepsilon/3), \tag{20}$$

for a fixed constant C. When $k > n_0$, applying Assumption 5 into the first term in (19) yields

$$\begin{aligned} &\left\| \sum_{i=1} \mathbf{G}^k L^{k-i}(a[i] - a[i-1]) \right\|_2 \\ &\leq \left\| \sum_{i=k-n_0}^{k} \mathbf{G}^{k-i}(a[i] - a[i-1]) \right\|_2 \\ &\quad + \left\| \sum_{i=1}^{k-n_0-1} \mathbf{G}^{k-i}(a[i] - a[i-1]) \right\|_2 \\ &< \left\| \sum_{i=k-n_0}^{k} \mathbf{G}^{k-i}(a[i] - a[i-1]) \right\|_2 + (\varepsilon/3). \end{aligned} \tag{21}$$

The boundedness of $\|\sum_{i=1}^{k-n_0-1} \mathbf{G}^{k-i}(a[i] - a[i-1])\|_2$ is easily proved by the component-wise triangle inequality (refer to Lemma 1). Define the notation s_k as

$$s_k = \max\{|a[i] - a[i-1]| \mid k - n_0 \leq i \leq k\}, \tag{22}$$

where k and n_0 satisfy Assumption 5. Let $\omega_o = (m/\Delta_t)$ where $0 < m < 2$ from the constraint (16). Then the vector term $\mathbf{G}^i$ can be generalized as

$$\mathbf{G}^i = \begin{bmatrix} \frac{i(i-1)}{2}(1-m)^{i-2}\Delta_t^2 \\ i((i-2)m+1)(1-m)^{i-2}\Delta_t \\ \left(\frac{(i-1)(i-2)}{2}m^2 + (i-2)m + 1\right)(1-m)^{i-2} \end{bmatrix}. \tag{23}$$

Since the choice of m in $0 < m < 2$ may change the sign of $(1 - m)$, we will consider two cases which depend on the sign of $(1 - m)$. The upper bound of sum in (21) will be approximated respectively for two cases, and it will be compared each other.

Case 1 ($0 < m \leq 1$): The radius of convergence is non-negative. Therefore, the row components G_1^i, G_2^i, and $G_3^i \geq 0$ for all $i \geq 0$. By applying the component-wise triangle inequality to the matrix power series term of (21), we have

$$\left\| \sum_{i=k-n_0}^{k} \mathbf{G}^{k-i}(a[i]-a[i-1]) \right\|_2 \leq s_k \left\| \sum_{i=0}^{n_0} \mathbf{G}^i \right\|_2 \leq s_k \left\| \sum_{i=0}^{k-1} \mathbf{G}^i \right\|_2. \tag{24}$$

By calculating the power series of (24), we can get

$$\begin{aligned} &\left\| \sum_{i=0}^{k-1} \mathbf{G}^i \right\|_2 \\ &= \left\| \sum_{i=0}^{k-1} \begin{bmatrix} \frac{i(i-1)}{2}(1-m)^{i-2}\Delta_t^2 \\ i((i-2)m+1)(1-m)^{i-2}\Delta_t \\ \left(\frac{(i-1)(i-2)}{2}m^2+(i-2)m+1\right)(1-m)^{i-2} \end{bmatrix} \right\|_2 \\ &= \left\| \begin{bmatrix} \Delta_t^2\left(\frac{1}{m^3}-\frac{(1-m)^k}{2}\left(\frac{k(k-1)}{m(1-m)^2}+\frac{2k}{m^2(1-m)}+\frac{2}{m^3}\right)\right) \\ \Delta_t\left(\frac{3}{m^2}-(1-m)^k\left(\frac{k(k-1)}{(1-m)^2}+\frac{k}{m(1-m)}+\frac{1}{m^2}\right)\right) \\ \frac{3}{m}-(1-m)^k\left(\frac{k(k-1)m}{2(1-m)^2}+\frac{2k-1}{1-m}+\frac{1}{m(1-m)}\right) \end{bmatrix} \right\|_2 \\ &\leq \left\| \begin{bmatrix} \frac{\Delta_t^2}{m^3} & \frac{3\Delta_t}{m^2} & \frac{3}{m} \end{bmatrix}^T \right\|_2. \end{aligned} \tag{25}$$

Case 2 ($1 < m < 2$): The radius $(1-m)$ of convergence is negative, and so raising the power of $(1-m)$ alters its sign. Therefore, applying the component-wise triangle inequality derives the upper bound of the sum in (21) as

$$\begin{aligned} &\left\| \sum_{i=k-n_0}^{k} \mathbf{G}^{k-i}(a[i]-a[i-1]) \right\|_2 \Big/ s_k \\ &\leq \left\| \sum_{i=0}^{k-1} \begin{bmatrix} \left\| \frac{i(i-1)}{2}(1-m)^{i-2}\Delta_t^2 \right\| \\ |i((i-2)m+1)(1-m)^{i-2}\Delta_t| \\ \left|\left(\frac{(i-1)(i-2)}{2}m^2+(i-2)m+1\right)(1-m)^{i-2}\right| \end{bmatrix} \right\|_2 \\ &\leq \left\| \sum_{i=0}^{k-1} \begin{bmatrix} \frac{i(i-1)}{2}(m-1)^{i-2}\Delta_t^2 \\ i((i-2)m+1)(m-1)^{i-2}\Delta_t \\ \left(\frac{(i-1)(i-2)}{2}m^2+(i-2)m+1\right)(m-1)^{i-2} \end{bmatrix} + \begin{bmatrix} 0 \\ 2\Delta_t \\ 2 \end{bmatrix} \right\|_2 \\ &\leq \left\| \begin{bmatrix} \frac{\Delta_t^2}{(2-m)^3} \Delta_t\left(2+\frac{3m-2}{(2-m)^3}\right) 2+\frac{3m^2-6m+4}{(2-m)^3} \end{bmatrix}^T \right\|_2 \end{aligned} \tag{26}$$

4.2 *Boundedness of estimation error*

The value of the upper bound on (25) is smaller than the one on (26), which means that the case for $1 < m < 2$ has larger error in a worst case. By this reason, we consider the case only for $0 < m \leq 1$. From the upper bound (25), $\|\sum_{i=0}^{k-1} G^i\|_2$ has the upper bound as:

$$\left\| \sum_{i=0}^{k-1} \mathbf{G}^i \right\|_2 \leq \sqrt{\frac{\Delta_t^4}{m^6}+\frac{9\Delta_t^2}{m^4}+\frac{9}{m^2}}. \tag{27}$$

Lemma 1 (general property of LESO's estimation error): *Define $0 < m \leq 1$. 2-norm sequence $\|e[k]\|_2$ of LESO's estimation error is bounded over all time $t \in [0, T]$ if the Assumption 4 holds.*

Proof: From the Assumption 3, we know that $|a[i]| \leq k_2$ for all i and a known constant k_2. The tracking estimation error $e[k]$ is obtained from (14). Thus, applying Cauchy's inequality and the triangle inequality to 2-norm sequence $\|e[k]\|_2$ yields

$$\begin{aligned} &\|e[k]\|_2 \\ &= \left\| (\mathbf{I}+\Delta_t(\mathbf{A}-\mathbf{LC}))^k e[0] + \sum_{i=1}^{k} \mathbf{G}^{k-i}(a[i]-a[i-1]) \right\|_2 \\ &\leq \|e[0]\|_2 + \Delta_t\|\mathbf{A}-\mathbf{LC}\|_2 \left\| \sum_{i=0}^{k-1} (\mathbf{I}+\Delta_t(\mathbf{A}-\mathbf{LC}))^i \right\|_2 \|e[0]\|_2 \\ &\quad + 2k_2 \left\| \sum_{i=0}^{k-1} \mathbf{G}^i \right\|_2. \end{aligned}$$

From (27), $\|\sum_{i=0}^{k-1} G^k\|_2$ is shown as bounded. $e[0]$ is bounded from Assumption 4. The 2-norm of $(\mathbf{A}-\mathbf{LC}))$ is upper-bounded by $\sqrt{2+(\Delta_t^6/m^6)+(9\Delta_t^4/m^4)+(9\Delta_t^2/m^2)}$. The 2-norm of the sum of $(\mathbf{I}+\Delta_t(\mathbf{A}-\mathbf{LC}))^i$ is bounded and it converges to the 2-norm of $(\mathbf{A}-\mathbf{LC}))^{-1}$ as $k \to \infty$ since the spectral radius of $(\mathbf{I}+\Delta_t(\mathbf{A}-\mathbf{LC}))$ is less than 1. Thus $\|e[k]\|_2$ is bounded over all time $t \in [0, T]$. □

Lemma 2 (boundedness of LESO's estimation error): *Define $0 < m \leq 1$. As $k > n_0$ under Assumptions 4 and 5, 2-norm sequence $\|e[k]\|_2$ of LESO's estimation error is upper bounded by*

$$s_k\sqrt{(\Delta_t^4/m^6)+(9\Delta_t^2/m^4)+(9/m^2)}+(2\varepsilon/3)$$

for any positive number ε.

Proof: The bound of $\|e[k]\|_2$ can be found as

$$\begin{aligned}
&\|e[k]\|_2 \\
&< \left\| \sum_{i=1}^{k} \mathbf{G}^{k-i}(a[i]-a[i-1]) \right\|_2 + (\varepsilon/3) && \text{(from (19))} \\
&< \left\| \sum_{i=k-n_0}^{k} \mathbf{G}^{k-i}(a[i]-a[i-1]) \right\|_2 + (2\varepsilon/3) && \text{(from (21))} \\
&< s_k \left\| \sum_{i=0}^{k-1} \mathbf{G}^{i} \right\|_2 + (2\varepsilon/3) && \text{(from (24))} \\
&< s_k \sqrt{\frac{\Delta_t^4}{m^6} + \frac{9\Delta_t^2}{m^4} + \frac{9}{m^2}} + (2\varepsilon/3), && \text{(from (27))}
\end{aligned}$$

where $k > n_0$ and $0 < m \leq 1$. □

Theorem 2 (convergence condition of LESO's estimation error)**:** *Let* $0 < m \leq 1$. *2-norm sequence* $\|e[k]\|_2$ *of LESO's estimation error converges to limit zero if, for any positive number* ε, $s_k\sqrt{(\Delta_t^4/m^6) + (9\Delta_t^2/m^4) + (9/m^2)} \leq (\varepsilon/3)$ *under Assumptions 4 and 5 as* $k > n_0$.

Proof: If $s_k\sqrt{(\Delta_t^4/m^6) + (9\Delta_t^2/m^4) + (9/m^2)} \leq (\varepsilon/3)$ under Assumptions 4 and 5, then it can be easily induced from Lemma 2 that $\|e[k]\|_2 < \varepsilon$ for any positive number ε as $k > n_0$. Thus, by the theorem of convergent sequence, it is proved. □

4.3 *Numerical solutions of tracking convergence condition*

Theorem 2 entails the induced result that LESO is globally asymptotically stable at an equilibrium point. To get more specific tracking convergence condition of observer bandwidth ω_o from Theorem 2, we need to solve the inequality problem as

$$s_k\sqrt{(\Delta_t^4/m^6) + (9\Delta_t^2/m^4) + (9/m^2)} \leq (\varepsilon/3). \tag{28}$$

Since both sides in (28) are non-negative, the problem finding the tracking convergent condition is reduced to solving the inequality problem as

$$(\Delta_t/m)^6 + 9(\Delta_t/m)^4 + 9(\Delta_t/m)^2 \leq (\Delta_t\varepsilon/3s_k)^2. \tag{29}$$

Let $x = (\Delta_t/m)^2$. Then the equality part in (29) becomes the cubic equality problem as

$$x^3 + 9x^2 + 9x = (\Delta_t\varepsilon/3s_k)^2, \tag{30}$$

where $x > 0$. By letting $x = p + (6/p) - 3$, (30) can be rewritten as

$$(p^3)^2 + \left(27 - (\Delta_t\varepsilon/3s_k)^2\right)p^3 + 216 = 0. \tag{31}$$

Equation (31) is in the form of a quadratic equation of p^3. Using the quadratic formula, the roots of p^3 are

$$\left(-\left(27 - \left(\frac{\Delta_t\varepsilon}{3s_k}\right)^2\right) \pm \sqrt{\left(27 - \left(\frac{\Delta_t\varepsilon}{3s_k}\right)^2\right)^2 - 864}\right) \Big/ 2. \tag{32}$$

Since $x > 0$ and (30) has one real root and one complex conjugate pair, the discriminant of (31) must be negative by choosing ε as

$$0 < \varepsilon < \left(3\sqrt{27 + 12\sqrt{6}}\right)(s_k/\Delta_t), \tag{33}$$

where $s_k > 0$. By calculating the roots for $\omega_o = (m/\Delta_t)$ from the roots of p by the backward calculation, the solution sets of (28) can be written as follows.

Let

$$\theta = \frac{1}{3}\arctan\left(\frac{\left(\sqrt{-\left(27 - \left(\frac{{}_t\varepsilon}{3s_k}\right)^2\right)^2 + 864}\right)}{\left(\left(\frac{{}_t\varepsilon}{3s_k}\right)^2 - 27\right)}\right). \tag{34}$$

Figure 1 shows graphically the places of the possible root $(\Delta_t/m) = (1/\omega_o)$ in (28). The roots exist if the curve passes through the quadrant where $(\Delta_t/m) > \Delta_t$ and $(\Delta_t\varepsilon/3s_k) > 0$.

Solution 1: As $0 < \varepsilon < (9\sqrt{3})(s_k/\Delta_t)$, the inequality of (28) holds if

$$\left(1/\sqrt{2\sqrt{6}\cos\theta - 3}\right) \leq \omega_o \leq (1/\Delta_t) \tag{35}$$

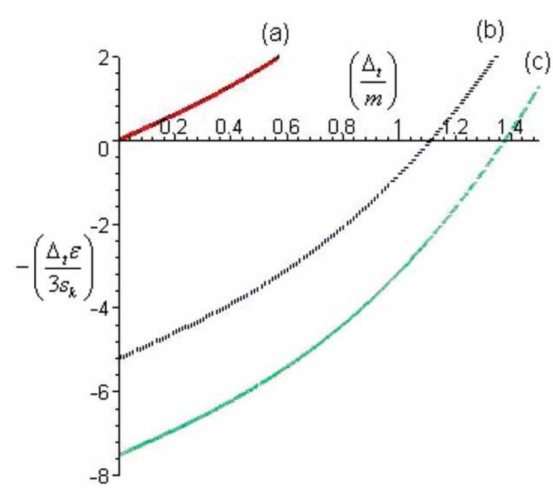

Figure 1. Graphical solutions in (Δ_t/m) – plane.

where $(\pi/6) \le \theta \le \arccos(\sqrt{6}/4)$. As the inequality holds, the curve line for equality part of (28) lies between (a)−(b) region in figure 1.

Solution 2: As $(9\sqrt{3})(s_k/\Delta_t) \le \varepsilon < (3\sqrt{27+12\sqrt{6}})(s_k/\Delta_t)$, the inequality of (28) holds if

$$\left(1/\sqrt{2\sqrt{6}\cos\theta - 3}\right) \le \omega_o \le (1/\Delta_t) \tag{36}$$

where $0 \le \theta \le (\pi/6)$. As the inequality holds, the curve line for equality part of (28) lies between (*b*)–(*c*) region in figure 1.

Corollary 1: *Let* $0 < m \le 1$. *As* $k > n_0$ *under the Assumptions 4 and 5, if there exists observer bandwidth* ω_o *satisfying one of solutions* (35) *and* (36) *for any positive number* ε, *then* 2-*norm sequence* $\|e[k]\|_2$ *of LESO's estimate error converges to limit zero.*

5. Optimal solution of fast tracking problem in discrete time

The description for the fast tracking problem (Problem 2) is modified in discrete time domain as in Problem 3.

Problem 3: Given any positive number ε, minimize t_0 such that $\|e(t)\|_2 < \varepsilon$, for all $k(=\lfloor t/\Delta_t \rfloor) > (\lfloor t_0/\Delta_t \rfloor)$ and $\Delta_t > 0$ subject to: discrete time system representation (14), one tuning parameter $0 < \omega_o \le (1/\Delta_t)$ known desired trajectory of the position $v(t)$ unknown function $f(\cdot)$ with Assumptions 1–5.

The feasible solutions of ω_o are specified in solutions (35) and (36). The angle θ has three factors, Δ_t, ε and s_k. The sampling time Δ_t is delimited by the characteristics of the hardware in use. The absolute error tolerance ε can be determined by a practical control engineer. Thus we assume both the sampling time Δ_t and the absolute error tolerance ε are fixed so that the only factor for the feasible solutions is s_k. If $s_k \gg 0$, then the fast tracking problem entirely lies above the curve figure 1, which means that no feasible solution exists. Note that we require $0 < \omega_o \le (m/\Delta_t)$ and $0 < m \le 1$ for feasibility. Thus feasible solutions lie on the portion of the quadrant where $(\Delta_t/m) \ge \Delta_t$ and $(\Delta_t\varepsilon/3s_k) > 0$. If $s_k \to 0$, then the curve on figure 1 moves as $(a) \to (b) \to (c)$. Therefore, the sampling time Δ_t touches the curve at first then the other values on (Δ_t/m) axis.

Theorem 3: *The numerical solutions* (35) *and* (36) *are the feasible solutions of the fast tracking problem in discrete time domain. Then the choice of* $\omega_o = (1/\Delta_t)$ *is the optimal fast tracking observer bandwidth.*

6. Simulation

The practical model of the simulation in this section is used in Gao *et al.* (2001b, 2003) and Yoo (2005). The estimated mathematical model of motion control system is

$$\ddot{y} = (-1.41\dot{y} + 23.2T_d) + (23.2 - 40)u + 40u = f + 40u,$$

where y is the output position, u is the control voltage, and T_d is the torque disturbance. The corresponding linear extended state observer is

$$\dot{z}(t) = \begin{bmatrix} -3\omega_o & 1 & 0 \\ -3\omega_o^2 & 0 & 1 \\ -\omega_o^3 & 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 40 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 3\omega_o \\ 3\omega_o^2 \\ \omega_o^3 \end{bmatrix} y(t)$$

where the controller is defined as $u = (u_0 - z_3)/40$, $u_0 = k_p(r - z_1) - k_d z_2$, and r is reference input, $k_d = 2\xi\omega_c$, $\xi = 1$, and $k_p = \omega_c^2$, and the estimates should satisfy the conditions as

$$z_1 \to y,\ z_2 \to \dot{y}, \quad \text{and} \quad z_3 \to f \quad \text{as } t \to \infty.$$

The design objective is to rotate the load one revolution in one second with no overshoot and the control signal u has the physical constraint $|u| < 3.5$ volt.

The simulation is performed by Simulink using ode1 (Euler) with a fixed step of 1 ms sampling time. A step torque disturbance of 10% of the maximum torque is added at $t = 2$ second. Two different velocity profiles shown in figure 2 are used to generate the reference input for the simulation. In all our simulations, we used the fixed closed-loop bandwidth, $\omega_c = 100$(rad/s). Because ω_c was not considered on the analysis of estimation error, and also ω_c does not affect significantly the speed of the tracking estimation error (Yoo 2005). In the first simulation shown at figure 3, two different values of the observer bandwidth are used with the trapezoidal profile. They are $\omega_o = 100$ (rad/s) and $\omega_o = 1000$ (rad/s). As we expected, the simulations with $\omega_o = 1000$ (rad/s) shows more faster tracking tendency than the one with $\omega_o = 100$ (rad/s). The apparent difference can be seen on the tracking results about uncertain dynamics. From now on, we call 2-norm of estimate error an absolute error. The next simulation is performed with the two profiles where the value of the observer bandwidth, ω_o, is changed from 20 (rad/s) to 1000 (rad/s) and the results of the computed absolute error over those observer bandwidths are shown in figure 4.

The trapezoidal profile is used in figure 4(a), and the smooth S-curve profile in figure 4(b). As the value of observer bandwidth is getting smaller close to 20 (rad/s), the radius of convergence in (15) goes close to 1. Consequently it requires a long time to reduce the effect of estimate error. Since the effect of estimate error is

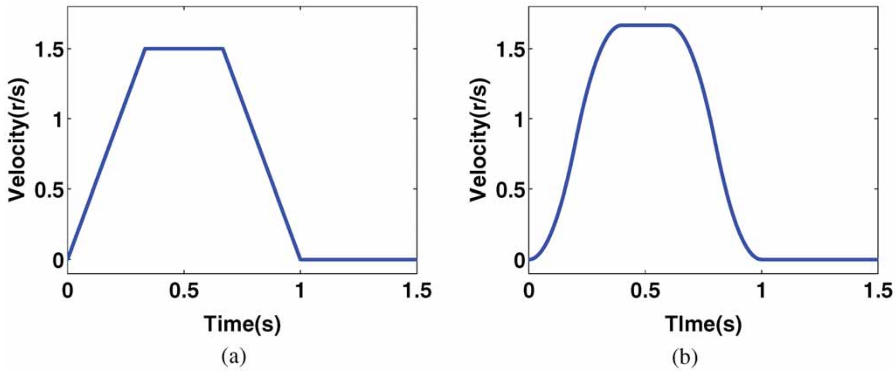

Figure 2. Velocity profiles: (a) trapezoidal profile; (b) smooth S-curve profile.

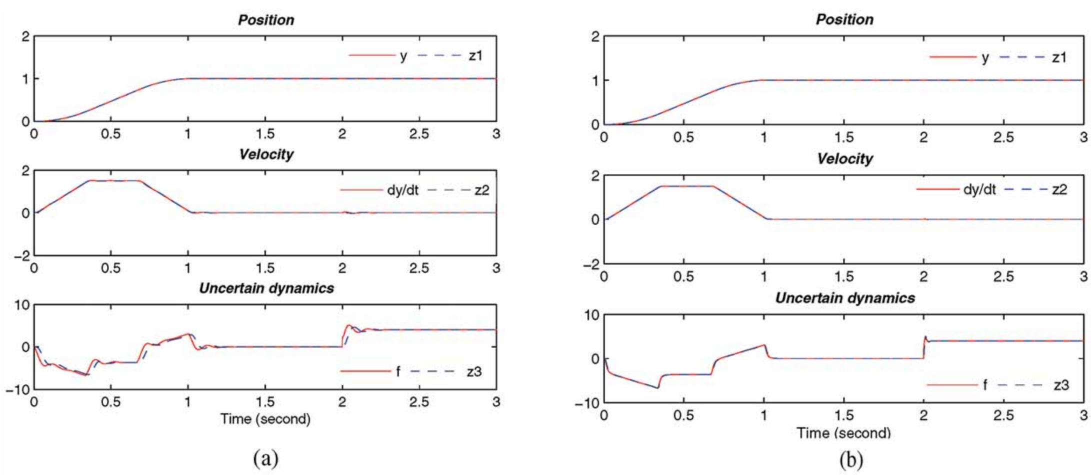

Figure 3. Tracking behaviours with trapezoidal profiles: (a) $\omega_c = 100\,(\text{rad/s})$ and $\omega_o = 100\,(\text{rad/s})$; (b) $\omega_c = 100\,(\text{rad/s})$ and $\omega_o = 1000\,(\text{rad/s})$.

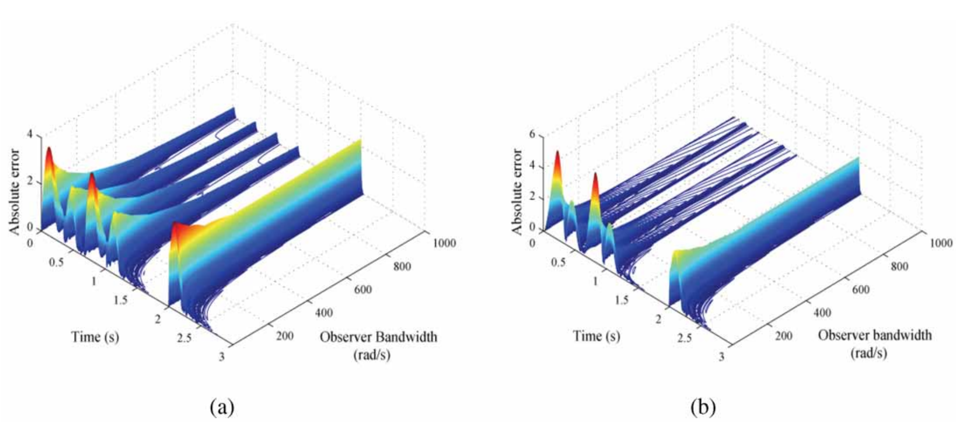

Figure 4. Absolute errors on the fixed $\omega_c = 100\,(\text{rad/s})$: (a) absolute error with the trapezoidal profile; (b) absolute errors with smooth the S-curve profile.

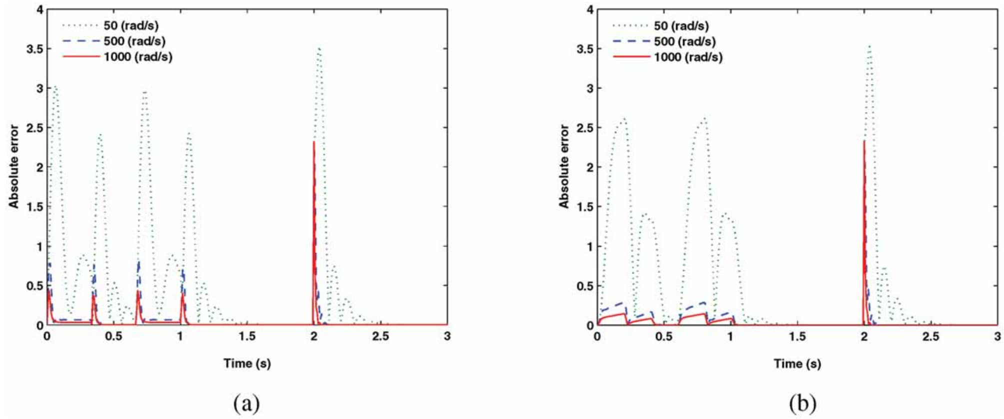

Figure 5. Comparison of absolute errors on the selected observer bandwidth ($\omega_o = 100, 500, 1000$ (rad/s) and $\omega_c = 100$ (rad/s)): (a) Comparison of absolute errors in the trapezoidal profile; (b) Comparison of absolute errors in the smooth S-curve profile.

accumulated as seen in (14), it incurs the large value of absolute error near the observer bandwidth 20 (rad/s). Conversely, as the value of the observer bandwidth is getting close to 1000 (rad/s), the radius of convergence goes close to zero. That means that it can reduce the effect of estimate error faster. So, the value of absolute error is getting smaller as shown in figure 4. For a better view of the comparison among the different values of observer bandwidth, three values, $\omega_o = 50, 500, 1000$ (rad/s), are selected. The results are shown in figure 5. As the observer bandwidth is 1000 (rad/s), its absolute error is smaller than the others, and the absolute error is decayed faster. For a comparison between two profiles, the observer bandwidth is fixed at 500 (rad/s) as shown in figure 6. When the trapezoidal profile is used, it has relatively higher absolute errors. Because the trapezoidal profile has larger jerk than one for the smooth S-curve profile, and one of the factors affecting s_k in (22) is the jerk of the trajectory function. This can easily be deduced from (1). Thus, the case with the smooth S-curve profile can reduce the effect of estimate error faster than the one with the trapezoidal profile.

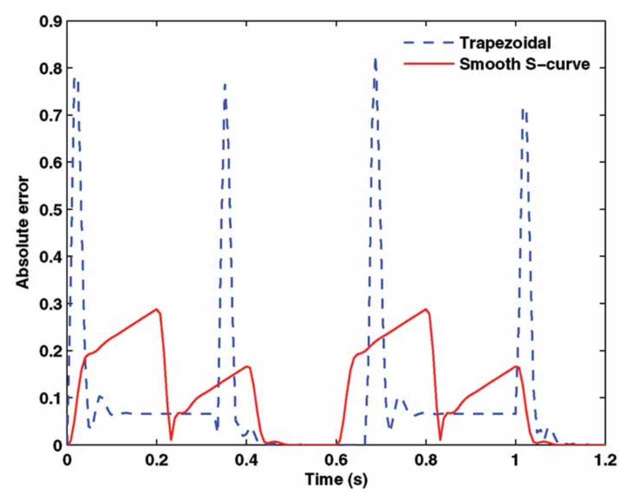

Figure 6. Comparison of absolute errors between trapezoidal profile and smooth S-curve profile in the fixed observer bandwidth $\omega_o = 500$ (rad/s) ($\omega_c = 100$ (rad/s)).

7. Conclusion

The fast tracking problem in discrete time domain has been constructed for the tracking error dynamics of LESO. The existence of the feasible solution of the fast tracking problem is strongly concerned with the uncertainty of motion control applications. To reduce the effects of the uncertainty, the selection of observer bandwidth and the design of motion profile must be taken into consideration. As shown in simulation results, the motion profile with smaller jerk is desirable to the fast tracking estimation of observer. We have also shown that the observer bandwidth takes the central role of tracking estimation of LESO. With digital implementation in LESO, the observer bandwidth in the feasible solution of the fast tracking problem takes values less than the inverse of sampling time. The optimal fast tracking observer bandwidth is the inverse of sampling time. These results provide a design guideline for the LESO design in motion control applications.

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