Analyzing Control System Robustness

Dan Simon, d.j.simon@csuohio.edu
Cleveland State University
Department of Electrical Engineering
Stilwell Hall Room 332
1960 E. 24th Street
Cleveland, OH 44115

When an engineer designs a control system, the design is usually based on some mathematical model for the system to be controlled. However, the system model is only an approximation. In reality the system may behave differently than the model indicates, or the system parameters may vary with time. In practice, though, control systems that are designed on the basis of some system model often work quite well.

A control system designer (and the designer's manager and customer) may have nagging doubts about how much the real system can depart from the model before the control system performance fails to meet its requirements or even becomes unstable. The Nyquist stability test was developed in 1932 as a graphical method of determining the stability of a linear single-input-singleoutput (SISO) feedback control system. This test shows how robust a control system is to system uncertainties. That is, given a stable control system, how much can the plant gain change before the system becomes unstable (i.e., what are the *gain margins* of the system)? How much can the phase shift change before the system becomes unstable (i.e., what is the *phase margin* of the system)?

The Nyquist stability criterion continues to be useful, but most real control systems are multiple-input-multiple-output (MIMO) and hence are not amenable to robustness analysis via the Nyquist criterion. Perturbation analysis based on singular values (SVs) was initially developed by George Zames in 1966. SV analysis is becoming popular as a general way of analyzing the stability robustness of MIMO systems. Also, the Nyquist stability criterion can be used only with specific types of perturbations (gain and phase perturbations) while SV analysis can be applied in the presence of any type of perturbation. This paper presents SV analysis as a generalization of the Nyquist criterion.

The Nyquist Stability Criterion

Most texts on classical control discuss the Nyquist stability criterion. Many engineers (especially electrical and mechanical engineers) are exposed to this topic in their senior-level controls class. This section summarizes and illustrates the criterion with two general examples.

Suppose that we are given a SISO control system with input $u(t)$, output $y(t)$, plant $G(s)$, and feedback controller $K(s)$ (where s is the Laplace transform variable) as depicted in Figure 1. Further assume that N_p is the number of poles of $K(s)G(s)$ (i.e., the open loop transfer function) in the closed right half plane. The Nyquist stability criterion states that the closed loop control system is stable if and only if the image of a closed contour encircling (in the clockwise direction) the right half s-plane as mapped through $K(s)G(s)$ encircles the point -1 exactly N_p times in the counterclockwise direction. Note that 1 encirclement in the clockwise direction is considered as -1 encirclement in the counterclockwise direction.

The Nyquist criterion can be used to determine stability robustness to unmodelled gain and phase shifts. Consider the original block diagram of Figure 1 modified to include an uncertain gain g and an uncertain phase shift θ. We can assume without loss of generality that g is a real number greater than 0 and θ is a real number between 0 and 2π. This uncertain system is depicted in Figure 2. In the nominal case we have $g = 1$ and $\theta = 0$. The gain g multiplies the open loop transfer function of the nominal system. So each point on the Nyquist plot of the perturbed system is g times the corresponding point on the Nyquist plot of the nominal system. The gain g then simply scales the nominal Nyquist plot. Similarly, the phase shift θ multiplies each point on the nominal Nyquist plot causing a rotation of the nominal Nyquist plot through an angle θ about the origin.

Example 1 – Consider the uncertain system of Figure 2 where

$$K(s)G(s) = \frac{-0.5(s-1)}{2s+1} \tag{1}$$

The Nyquist plot is given in Figure 3. The open loop transfer function $K(s)G(s)$ has zero right half plane poles, so the Nyquist criterion states that the closed loop system is stable if and only if the Nyquist plot encircles the point -1 zero times in the counterclockwise direction. It can be seen from Figure 3 that the plot does indeed encircle the point -1 zero times, indicating that the nominal closed loop system is stable. It can also be seen that any $g < 1$ will contract the Nyquist plot and that the system will therefore remain stable for any $g < 1$. We see in addition that if $g > 2$, there exists a θ (namely $\theta = \pi$) such that the Nyquist plot of the perturbed system will encircle the -1 point one time in the clockwise direction, which is equivalent to -1 encirclement in the counterclockwise direction. But for any $g < 2$, the Nyquist plot will not encircle the -1 point regardless of the phase shift θ. We conclude from these observations that the closed loop system is stable for all θ if $g < 2$.

It can be deduced from this example that, in general, a stable feedback system with a Nyquist plot that encircles the point -1 zero times will remain stable for all θ if and only if

$$g < \frac{1}{\max_\omega |K(j\omega)G(j\omega)|} \tag{2}$$

Example 2 – Consider the uncertain system of Figure 2 where

$$K(s)G(s) = \frac{-4(s+1)}{s-2} \tag{3}$$

The Nyquist plot is given in Figure 4. The open loop transfer function $K(s)G(s)$ has one right half plane pole (located at $s = 2$), so the Nyquist criterion states that the closed loop system is stable if and only if the Nyquist plot encircles the point -1 one time in the counterclockwise direction. It can be seen from Figure 4 that the plot does indeed encircle the point -1 one time, indicating that the nominal closed loop system is stable. It can also be seen that any $g > 1$ will expand the Nyquist plot and that the system will therefore remain stable for any $g > 1$. We see in addition that if $g < 0.5$, there exists a θ (namely $\theta = \pi$) such that the Nyquist plot will no longer encircle the -1 point. But for any $g > 0.5$, the Nyquist plot will encircle the -1 point regardless of the phase shift θ. We conclude from these observations that the closed loop system is stable for all θ if $g > 0.5$.

It can be deduced from this example that, in general, a stable feedback system with a Nyquist plot that encircles the point -1 one time will remain stable for all θ if and only if

$$g > \frac{1}{\min_\omega |K(j\omega)G(j\omega)|} \tag{4}$$

The Small Gain Theorem

In 1966 George Zames developed the small gain theorem to analyze the stability of MIMO control systems in the presence of unstructured perturbations (i.e., perturbations that are bounded in some sense but that do not have any special structure). Before we state the small gain theorem, we need to establish a couple of definitions.

Definition 1 *Consider an m-input, n-output transfer function matrix* $G(s)$ *(an* $n \times m$ *matrix). Let* $p = \min(m, n)$*. The singular values (SVs) of* $G(s)$ *are defined as*

$$\sigma_i[G(s)] = \sqrt{\lambda_i[G^T(s)G(s)]}\ (i = 1, \ldots, p) \tag{5}$$

where $\lambda_i[\cdot]$ *indicates the eigenvalues of a matrix. Note that the singular values are functions of s (the Laplace transform variable). The singular values are always nonnegative and are usually arranged in descending order, so* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$.

Definition 2 *Consider an m-input, n-output transfer function matrix* $G(s)$ *(an* $n \times m$ *matrix). The infinity-norm of* $G(j\omega)$ *is defined as*

$$\|G(j\omega)\|_\infty = \max_\omega \sigma_1[G(j\omega)] \tag{6}$$

In words, the infinity-norm of a transfer function is equal to the largest singular value of the transfer function taken over all frequencies.

Now we are in a position to state the small gain theorem.

Theorem 1 *The system shown in Figure 5 is stable for all* $\Delta(s)$ *with* $\|\Delta(j\omega)\|_\infty \leq 1$ *if and only if* $M(s)$ *is stable and* $\|M(j\omega)\|_\infty < 1$*. In this case the system is said to be robustly stable.*

Note that in the above theorem $M(s)$ and $\Delta(s)$ can (in general) be MIMO transfer functions. There are no limitations on the structure or form of the perturbation $\Delta(s)$ except for the magnitude of its infinity norm; therefore, $\Delta(s)$ is referred to as an *unstructured perturbation*.

Recall from Figure 2 the general SISO feedback system that the Nyquist plot was used to analyze. This figure can be redrawn as in Figure 6. Since this is a SISO system $\Delta(s)$ is a scalar transfer function. The infinity-norm of a scalar transfer function is simply equal to its maximum magnitude (taken over all frequencies). So restricting $\|\Delta(j\omega)\|_\infty \leq 1$ for a SISO system is equivalent to restricting $|\Delta(j\omega)| \leq 1$ over all frequencies while allowing its phase to be arbitrary. So the small gain theorem, when applied to Figure 6, states that the system is stable for all $\Delta(s)$ with $|\Delta(j\omega)|_{\max} \leq 1$ if and only if $K(s)G(s)ge^{j\theta}$ is stable and

$$|K(j\omega)G(j\omega)ge^{j\theta}| < 1 \text{ for all } \omega \tag{7}$$

Recalling that g is a real number greater than 0, this can be restated as

$$g < \frac{1}{\max_\omega |K(j\omega)G(j\omega)|} \tag{8}$$

This requirement is exactly the same as the Nyquist stability criterion of (2) for a system that does not have any open loop poles in the right half plane. So if (8) is applied to Example 1, we obtain the stability requirement that $g < 1/0.5 = 2$, which is exactly the same as that derived by the Nyquist criterion.

Now for a system that has one open loop pole in the right half plane (as in Example 2) the small gain theorem makes the same claim as above: the system is robustly stable for all $\Delta(s)$ with $|\Delta(j\omega)|_{\max} \leq 1$ if and only if $K(s)G(s)ge^{j\theta}$ is stable and $g < 1/|K(j\omega)G(j\omega)|_{max}$. But the Nyquist criterion (4) says that a system that has one open loop pole in the right half plane is stable for $g > 1/|KG|_{min}$. Whence the discrepancy?

The apparent discrepancy can be explained by noting that the small gain theorem requires that the open loop system be stable. But if the open loop system has a pole in the right half plane then it is not stable. So the small gain theorem says that a system like that illustrated in Example 2 is not robustly stable since its open loop transfer function is not stable. This is actually consistent with the Nyquist criterion because the only requirement on the perturbation $\Delta(s)$ in Figure 6 is that $|\Delta(j\omega)|_{\max} \leq 1$. That means that $|\Delta(j\omega)|$ can be made arbitrarily small. So if the Nyquist plot of $K(s)G(s)$ encircles the -1 point one time, then an arbitrarily small value of $|\Delta(j\omega)|$ will contract the Nyquist plot and result in zero encirclements of the -1 point, which indicates a loss of stability.

Hence both the Nyquist criterion and the small gain theorem agree that it cannot be said that the system of Figure 6 is robustly stable for all $\Delta(s)$ with $\|\Delta(j\omega)\|_\infty \leq 1$. This is interesting because even though the small gain theorem is a generalization of the Nyquist stability criterion, for this particular class of systems the Nyquist criterion gives more stability information than does the small gain theorem. The small gain theorem is too general in this case. This leads us to the structured singular value.

The Structured Singular Value

The structured singular value (SSV) can be used to determine the stability of a system that is subject to structured perturbations. So in addition to the small gain theorem requirement that the perturbation $\Delta(s)$ satisfy the inequality $\|\Delta(j\omega)\|_\infty \leq 1$, we also restrict the perturbation to a set of allowable perturbations denoted as $\bar{\Delta}$. For instance, $\bar{\Delta}$ could include the set of all real matrices, or the set of all block diagonal matrices with some specific structure. We use the notation $|\cdot|$ for the determinant of a matrix, and $\sigma_1(\cdot)$ for the largest singular value of a matrix.

Definition 3 *The SSV of a transfer function $M(s)$ with respect to the set $\bar{\Delta}$ of allowable perturbations is defined as*

$$\mu_{\bar{\Delta}}(M) = \frac{1}{\min_{\Delta \in \bar{\Delta}} \sigma_1(\Delta) \textit{ such that } |I - M\Delta| = 0} \tag{9}$$

unless no $\Delta \in \bar{\Delta}$ makes $|I - M\Delta| = 0$, in which case $\mu_{\bar{\Delta}}(M) = 0$. Note that $\mu_{\bar{\Delta}}(M)$ is a function of frequency ω.

Theorem 2 *The system shown in Figure 5 is internally stable for all $\Delta(s) \in \bar{\Delta}$ with $\|\Delta(j\omega)\|_\infty \leq 1$ if and only if $M(s)$ is stable and $\max_\omega \mu_{\bar{\Delta}}(M) < 1$.*

This statement was originally referred to as the Small Mu Theorem by John Doyle in 1982, but is now more often referred to as the small gain theorem for structured perturbations. If this theorem is satisfied the system is said to be *robustly stable*. $\Delta(s)$ is referred to as a *structured perturbation* because it is restricted to the set $\bar{\Delta}(s)$.

Figure 2 shows the general SISO feedback system that the Nyquist plot was used to analyze. This figure can be redrawn as in Figure 7. In order to guarantee the equivalence of Figure 2 and Figure 7 for all $g \in (0, \infty)$ and for all $\theta \in [0, 2\pi)$ we must restrict Δ to be real and positive. So our set of allowable perturbations $\bar{\Delta}$ consists of all real positive numbers.

$$\bar{\Delta} = \{\text{Real Positive Numbers}\} \tag{10}$$

It can be shown by analyzing the block diagram of Figure 7 that the transfer function from $w_d(t)$ to $y_d(t)$ is

$$N_{ydwd} = \frac{KG(1-g)e^{j\theta}}{1+KGe^{j\theta}} \tag{11}$$

(The dependence of functions on s or ω is henceforth dropped for the sake of conciseness.) The small gain theorem for structured perturbations says that the system of Figure 7 is stable for all $\Delta \in \bar{\Delta}$ with $|\Delta| \leq 1$ if and only if the nominal system (with $\Delta = 0$) is stable and $\max_\omega \mu_{\bar{\Delta}}(N_{ydwd}) < 1$.

In order to compute the SSV $\mu_{\bar{\Delta}}(N_{ydwd})$ from (9), we need to characterize the solutions of $|I - N_{ydwd}\Delta| = 0$. Since we are working with a SISO system, $|I - N_{ydwd}\Delta| = 1 - N_{ydwd}\Delta$. The roots of this equation can be solved using (11) as

$$\Delta = \frac{1+KGe^{j\theta}}{KG(1-g)e^{j\theta}} \tag{12}$$

Now if we realize that $KG(j\omega)$ in the above equation has a magnitude and a phase and can thus be written as $KG = Re^{j\phi}$, we can derive

$$\Delta = \frac{\cos(\theta+\phi) - j\sin(\theta+\phi) + R}{R(1-g)} \tag{13}$$

But recall that Δ must be a real number. So the coefficient of j in the above equation must be zero, which implies that $\theta + \phi = k\pi$, ($k = 0, \pm 1, \pm 2, \ldots$). This gives the solution

$$\Delta = \frac{|KG| \pm 1}{|KG|(1-g)} \tag{14}$$

There are two cases to consider in order to proceed further. The first case is for $|KG| < 1$ for all ω, which corresponds to Example 1 earlier in this paper. The second case is for $|KG| > 1$ for all ω, which corresponds to Example 2 earlier in this paper. These two cases will be considered in turn.

Structured Singular Value Analysis for $|KG| < 1$

If $|KG| < 1$ for all ω then, recalling that $\Delta \geq 0$, we can write (14) as

$$\Delta = \begin{cases} \frac{|KG|+1}{|KG|(1-g)} & \text{if} \quad g < 1 \\ \frac{|KG|-1}{|KG|(1-g)} & \text{if} \quad g > 1 \end{cases} \tag{15}$$

Note that $g = 1$ does not need to be considered because the block diagram of Figure 7 reduces to a system with no gain uncertainty in this case. Recalling that the singular value of a scalar is equal to the magnitude of the scalar, we have

$$\sigma_1(\Delta) = \begin{cases} \frac{1+|KG|}{|KG|(1-g)} & \text{if} \quad g < 1 \\ \frac{1-|KG|}{|KG|(g-1)} & \text{if} \quad g > 1 \end{cases} \tag{16}$$

Now recalling from (9) the definition of the SSV, we have

$$\max_{\omega} \mu_{\bar{\Delta}}(N_{ydwd}) < 1 \Rightarrow \begin{cases} g & > & -1/|KG|_{\max} & \text{if} \quad g < 1 \\ g & < & 1/|KG|_{\max} & \text{if} \quad g > 1 \end{cases} \tag{17}$$

The first condition in the above equation ($g > -1/|KG|_{\max}$ if $g < 1$) is always satisfied because we restricted g to be a positive number (without loss of generality). The second condition in the above equation ($g < 1/|KG|_{\max}$ if $g > 1$) is identical to the Nyquist stability criterion for the system (2).

Structured Singular Value Analysis for $|KG| > 1$

If $|KG| > 1$ for all values of ω, and if $g > 1$, then from (14) it can be seen that the only solutions of $|I - N_{ydwd}\Delta| = 0$ occur at negative values of Δ. But the set of allowable perturbations $\bar{\Delta}$ consists of real positive numbers (10). This means that there is no $\Delta \in \bar{\Delta}$ such that $|I - N_{ydwd}\Delta| = 0$, which means that $\mu_{\bar{\Delta}}(N_{ydwd}) = 0$ (see (9) and following). So $\max_\omega \mu_{\bar{\Delta}}(N_{ydwd}) < 1$, which implies that the system of Figure 7 is stable for all $\Delta \in \bar{\Delta}$.

This agrees with the Nyquist criterion for systems for which $|KG| > 1$ for all ω. The Nyquist criterion from (4) says that $g > 1/|KG|_{\min}$ for stability. But if $g > 1$ and $|KG| > 1$ for all ω, then $g > 1/|KG|_{\min}$.

Now consider the case $g < 1$. Then, recalling that $\Delta \geq 0$ and that we want to find the smallest Δ that satisfies $|I - N\Delta| = 0$, we can write (14) as

$$\Delta = \frac{|KG| - 1}{|KG|(1 - g)} \tag{18}$$

Again we note that $g = 1$ does not need to be considered because the block diagram of Figure 7 reduces to a system with no gain uncertainty in this case. Using the fact that the singular value of a scalar is equal to the magnitude of the scalar, we have

$$\sigma_1(\Delta) = \frac{|KG| - 1}{|KG|(1 - g)} \tag{19}$$

Now recalling from (9) the definition of the SSV, we can derive

$$\max_{\omega} \mu_{\bar{\Delta}}(N_{ydwd}) < 1 \Rightarrow g > \frac{1}{|KG|_{\min}} \tag{20}$$

This condition is identical to the Nyquist stability criterion for the system (4).

Summary

This paper has presented mathematical connections between three stability criteria:

1. The Nyquist stability criterion for SISO systems;

2. The small gain theorem for unstructured perturbations in MIMO systems;

3. The structured singular value for structured perturbations in MIMO systems.

This connection can provide an intuitive understanding of MIMO stability analysis using singular values for those who have had prior exposure to the Nyquist stability criterion.

Singular value techniques have been extended beyond the stability analysis presented in this paper to performance analysis and robust control system design. This approach has been successfully used for many control problems, including a magnetically levitated train, helicopter control, aero-engine control, power conversion, active sound control, aircraft control, flexible space structure control, and distillation process control. In the twentieth century, control theory evolved from classical control to modern control to optimal control to intellligent control. It is expected that this evolution will continue and that robust control (using the singular value techniques discussed in this paper) will become more common in practice.

Read More About It

- Burl, J., Linear Optimal Control, Addison Wesley Longman, Menlo Park, California, 1999.

- Doyle, J., “Analysis of feedback systems with structured uncertainties,” IEE Proceedings, Part D, vol. 133, 1982, pp. 45–56.

- Doyle, J., Wall, J., and Stein, G., “Performance and robustness analysis for structured uncertainty,” IEEE Conference on Decision and Control, 1982, pp. 629–636.

- Nyquist, H., “Regeneration theory,” Bell Systems Technical Journal, vol. 11, 1932, pp, 126–147.

- Packard, A., and Doyle, J., "The complex structured singular value," Automatica, vol. 29, 1993, pp. 71–109.
- Skogestad, S., and Postlethwaite, I., Multivariable Feedback Control, John Wiley & Sons, New York, 2000.
- Zames, G., "On the input-output stability of time-varying nonlinear feedback systems, Part I," IEEE Transactions on Automatic Control, vol. AC-11, 1966, pp. 228–238.
- Zames, G., "On the input-output stability of time-varying nonlinear feedback systems, Part II," IEEE Transactions on Automatic Control, vol. AC-11, 1966, pp. 465–476.
- Zhou, K., Essentials of Robust Control, Prentice Hall, Upper Saddle River, New Jersey, 1998.

About the Author

Dr. Dan Simon is an Assistant Professor in the Electrical Engineering Department at Cleveland State University in Cleveland, Ohio. Before entering academia he worked in industry for 14 years in the aerospace, automotive, biomedical, and process control fields. His research interests include control theory, neural networks, and fuzzy logic.

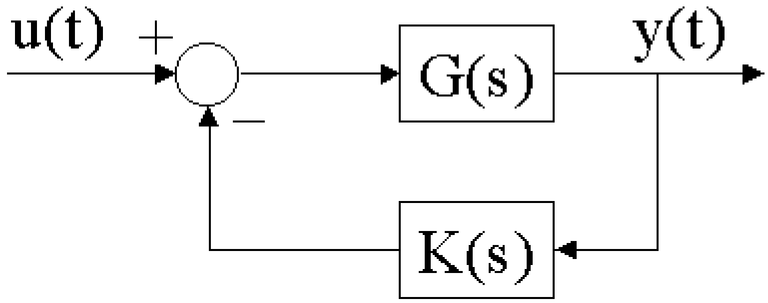

Figure 1: General Feedback Control System

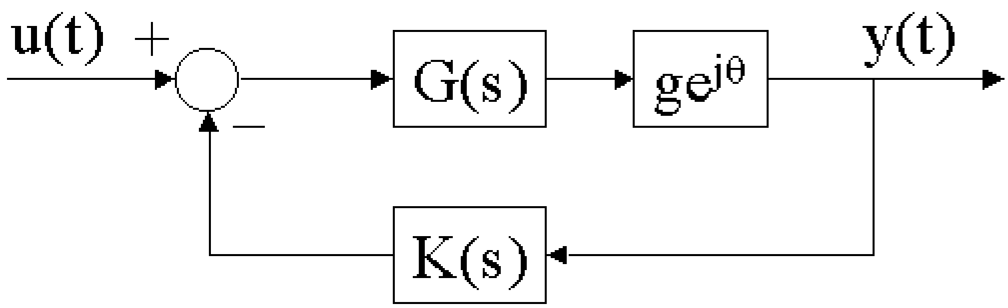

Figure 2: Feedback Control System with Gain and Phase Uncertainties

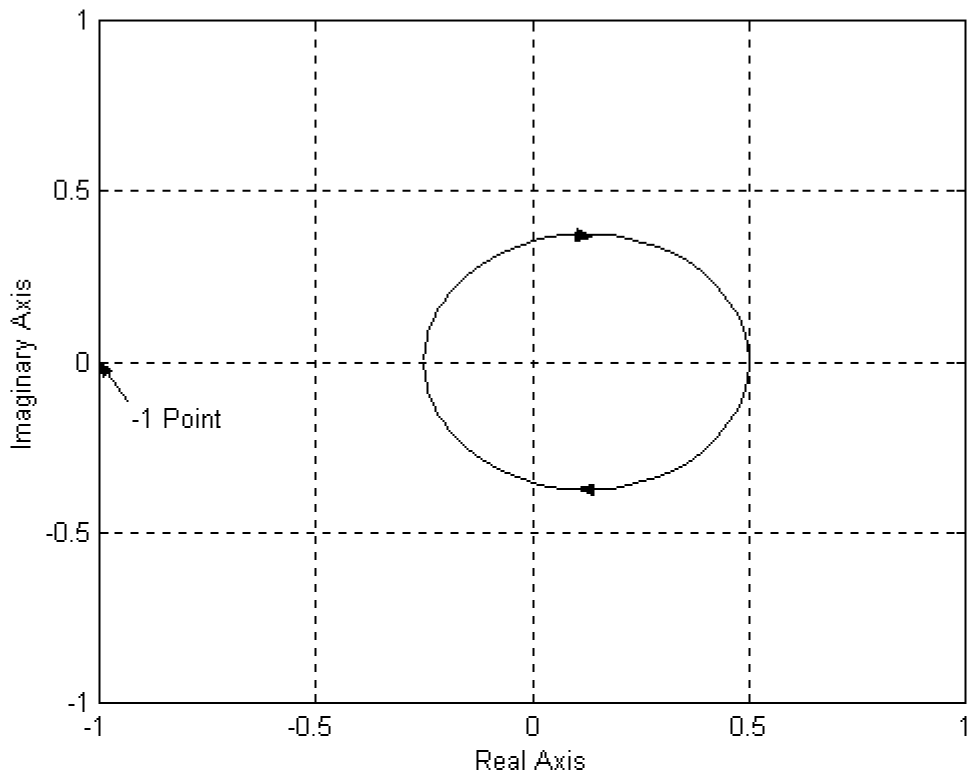

Figure 3: Nyquist Plot for Example 1

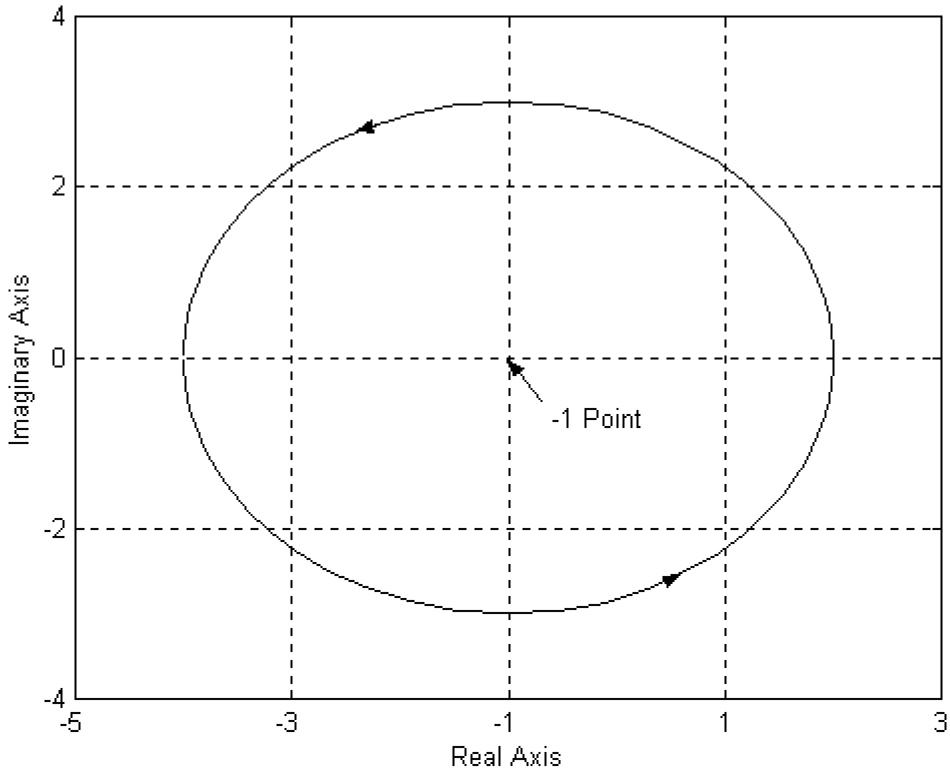

Figure 4: Nyquist Plot for Example 2

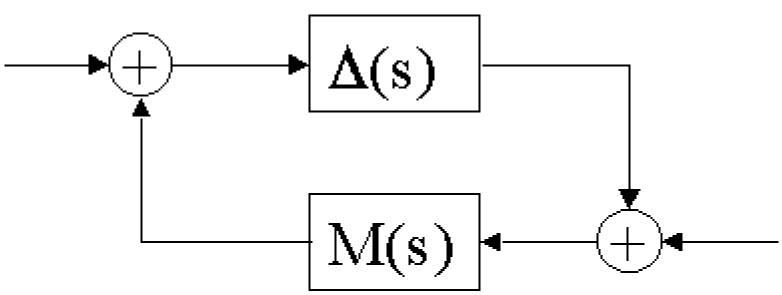

Figure 5: General Feedback Loop with Unstructured Perturbation

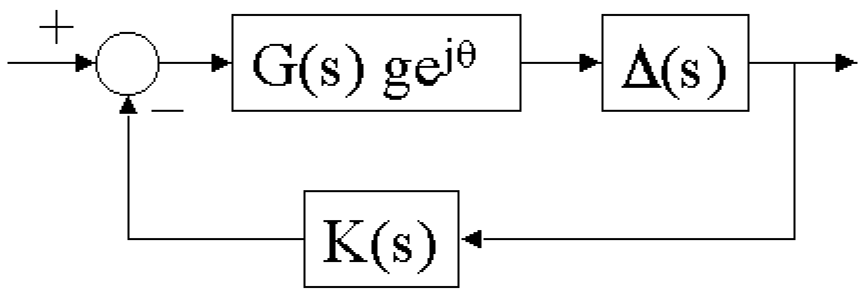

Figure 6: SISO Feedback Loop with Unstructured Perturbation

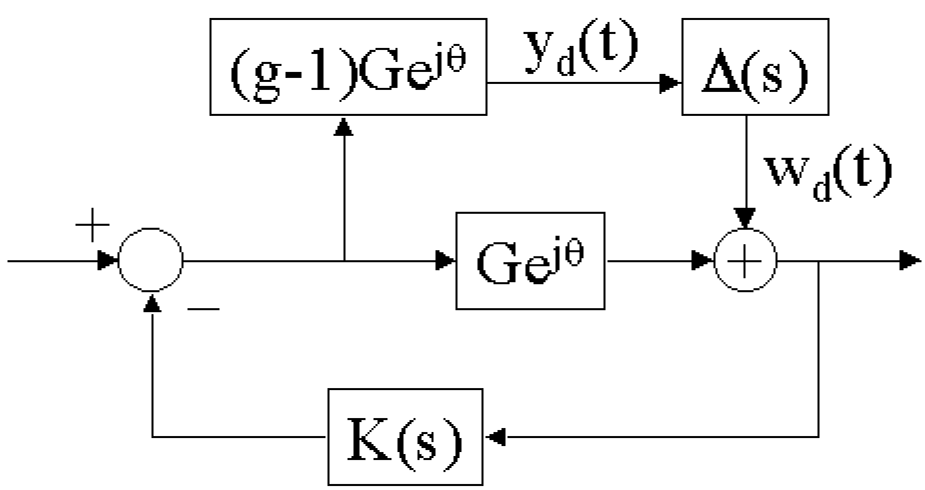

Figure 7: SISO Feedback Loop with Structured Perturbation