title:CDESO, current discrete estimator

abstract:Derivation of the current discrete estimator for various orders of ESO, the derivation is parameterized for the order of the ESO to simply resolve with various orders "showing the derivation work"

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date:4/28/4

    add solution for n=4

date:4/8/4

    clean up and verify results for different orders

date:4/7/4

    initial work
    work on deriving the transformations from matrix calculations
    there are difficulties with the matrix operations and matching sizes
    fix matrix size problems
    generalize for any order
    get results for 1-6 order
    try using the effective bandwidth modification for upping the order (the same results as final substitution are achieved)

```
Clear[ϕ, z, n, α]
```

■ CDESO derivation for n=3

Setting the value for the order "n" automatically derives "showing the work" of the a selected order

```
n = 3;
```

■ Solve for α_e in matrix form from the discrete pole location β

```
αz = (z - β)^n // Expand
```

$z^3 - 3 z^2 \beta + 3 z \beta^2 - \beta^3$

```
Clear[ϕ]
isoln = i → IdentityMatrix[n]
αz /. z → ϕ
% /. (β | β^y_) → β^y i
% /. x_ (ϕ | ϕ^y_) → x . ϕ^y
% /. ϕ^y_Integer → MatrixPower[ϕ, y]
αϕ = %
Dimensions[αϕ]
```

i → {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

$-\beta^3 + 3\beta^2\phi - 3\beta\phi^2 + \phi^3$

$-i\beta^3 + 3 i\beta^2\phi - 3 i\beta\phi^2 + \phi^3$

$-i\beta^3 + \phi^3 + (-3 i\beta).\phi^2 + (3 i\beta^2).\phi$

$-i\beta^3 + (-3 i\beta)$.MatrixPower[ϕ, 2] + $(3 i\beta^2).\phi$ + MatrixPower[ϕ, 3]

$-i\beta^3 + (-3 i\beta)$.MatrixPower[ϕ, 2] + $(3 i\beta^2).\phi$ + MatrixPower[ϕ, 3]

{4}

■ Solve for ϕ

```
Asoln = A → Table[If[i == j - 1, 1, 0], {i, n}, {j, n}];
Asoln // MatrixForm
```

A → {{0, 1, 0}, {0, 0, 1}, {0, 0, 0}}

```
Sum[MatrixPower[A, i] T^i / i!, {i, 0, n - 1}]
ϕsoln = ϕ → % /. Asoln
ϕ /. ϕsoln // MatrixForm
```

MatrixPower[A, 0] + T MatrixPower[A, 1] + $\frac{1}{2}$ T^2 MatrixPower[A, 2]

$\phi \to \left\{\left\{1, T, \frac{T^2}{2}\right\}, \{0, 1, T\}, \{0, 0, 1\}\right\}$

$$\begin{pmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}$$

■ Solve the expanded $\alpha\phi$

```
Dimensions[αϕ]
αϕ
αϕsoln = αϕ /. {isoln, ϕsoln};
αϕsoln // MatrixForm
αϕsoln // Dimensions
```

{4}

$-i\,\beta^3 + (-3\,i\,\beta).\text{MatrixPower}[\phi, 2] + (3\,i\,\beta^2).\phi + \text{MatrixPower}[\phi, 3]$

$$\begin{pmatrix} 1-3\beta+3\beta^2-\beta^3 & 3T-6T\beta+3T\beta^2 & \frac{9T^2}{2}-6T^2\beta+\frac{3T^2\beta^2}{2} \\ 0 & 1-3\beta+3\beta^2-\beta^3 & 3T-6T\beta+3T\beta^2 \\ 0 & 0 & 1-3\beta+3\beta^2-\beta^3 \end{pmatrix}$$

{3, 3}

■ Solve for L_c

Define the h row and column vectors

```
h = Table[If[j == 1, 1, 0], {j, n}]
hr = Reverse[h]
```

{1, 0, 0}

{0, 0, 1}

```
M = Table[h . MatrixPower[ϕ /. ϕsoln, i], {i, n}];
M // MatrixForm
```

$$\begin{pmatrix} 1 & T & \frac{T^2}{2} \\ 1 & 2T & 2T^2 \\ 1 & 3T & \frac{9T^2}{2} \end{pmatrix}$$

```
αϕsoln . Inverse[M] . hr;
% // FullSimplify
% // MatrixForm
```

$$\left\{1-\beta^3,\ \frac{3(-1+\beta)^2(1+\beta)}{2T},\ -\frac{(-1+\beta)^3}{T^2}\right\}$$

$$\begin{pmatrix} 1-\beta^3 \\ \frac{3(-1+\beta)^2(1+\beta)}{2T} \\ -\frac{(-1+\beta)^3}{T^2} \end{pmatrix}$$

■ CDESO derivation for n=4

Setting the value for the order "n" automatically derives "showing the work" of the a selected order

In[1116]:=
```
n = 4;
```

■ Solve for α_e in matrix form from the discrete pole location β

In[1117]:=
```
αz = (z - β)^n // Expand
```

Out[1117]=
$z^4 - 4 z^3 \beta + 6 z^2 \beta^2 - 4 z \beta^3 + \beta^4$

In[1118]:=
```
Clear[ϕ]
isoln = i → IdentityMatrix[n]
αz /. z → ϕ
% /. (β | β^y_) → β^y i
% /. x_ (ϕ | ϕ^y_) → x . ϕ^y
% /. ϕ^y_Integer → MatrixPower[ϕ, y]
αϕ = %
Dimensions[αϕ]
```

Out[1119]=
i → {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

Out[1120]=
$\beta^4 - 4 \beta^3 \phi + 6 \beta^2 \phi^2 - 4 \beta \phi^3 + \phi^4$

Out[1121]=
$i \beta^4 - 4 i \beta^3 \phi + 6 i \beta^2 \phi^2 - 4 i \beta \phi^3 + \phi^4$

Out[1122]=
$i \beta^4 + \phi^4 + (-4 i \beta).\phi^3 + (6 i \beta^2).\phi^2 + (-4 i \beta^3).\phi$

Out[1123]=
i β^4 + (-4 i β).MatrixPower[ϕ, 3] + (6 i β^2).MatrixPower[ϕ, 2] + (-4 i β^3).ϕ + MatrixPower[ϕ, 4]

Out[1124]=
i β^4 + (-4 i β).MatrixPower[ϕ, 3] + (6 i β^2).MatrixPower[ϕ, 2] + (-4 i β^3).ϕ + MatrixPower[ϕ, 4]

Out[1125]=
{5}

■ Solve for ϕ

In[1126]:=
```
Asoln = A → Table[If[i == j - 1, 1, 0], {i, n}, {j, n}];
Asoln // MatrixForm
```

Out[1127]//MatrixForm=
A → {{0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}, {0, 0, 0, 0}}

In[1128]:=

```
Sum[MatrixPower[A, i] T^i / i!, {i, 0, n - 1}]
ϕsoln = ϕ → % /. Asoln
ϕ /. ϕsoln // MatrixForm
```

Out[1128]=

$$\text{MatrixPower}[A, 0] + T\,\text{MatrixPower}[A, 1] + \frac{1}{2} T^2\,\text{MatrixPower}[A, 2] + \frac{1}{6} T^3\,\text{MatrixPower}[A, 3]$$

Out[1129]=

$$\phi \to \left\{\left\{1, T, \frac{T^2}{2}, \frac{T^3}{6}\right\}, \left\{0, 1, T, \frac{T^2}{2}\right\}, \{0, 0, 1, T\}, \{0, 0, 0, 1\}\right\}$$

Out[1130]//MatrixForm=

$$\begin{pmatrix} 1 & T & \frac{T^2}{2} & \frac{T^3}{6} \\ 0 & 1 & T & \frac{T^2}{2} \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

■ Solve the expanded $\alpha\phi$

In[1131]:=

```
Dimensions[αϕ]
αϕ
αϕsoln = αϕ /. {isoln, ϕsoln};
αϕsoln // MatrixForm
αϕsoln // Dimensions
```

Out[1131]=

{5}

Out[1132]=

$$i\,\beta^4 + (-4\,i\,\beta).\text{MatrixPower}[\phi, 3] + (6\,i\,\beta^2).\text{MatrixPower}[\phi, 2] + (-4\,i\,\beta^3).\phi + \text{MatrixPower}[\phi, 4]$$

Out[1134]//MatrixForm=

$$\begin{pmatrix} 1 - 4\beta + 6\beta^2 - 4\beta^3 + \beta^4 & 4T - 12T\beta + 12T\beta^2 - 4T\beta^3 & 8T^2 - 18T^2\beta + 12T^2\beta^2 - 2T^2\beta^3 & \frac{32T^3}{3} - 18T^3\beta \\ 0 & 1 - 4\beta + 6\beta^2 - 4\beta^3 + \beta^4 & 4T - 12T\beta + 12T\beta^2 - 4T\beta^3 & 8T^2 - 18T^2\beta + \\ 0 & 0 & 1 - 4\beta + 6\beta^2 - 4\beta^3 + \beta^4 & 4T - 12T\beta + \\ 0 & 0 & 0 & 1 - 4\beta + 6 \end{pmatrix}$$

Out[1135]=

{4, 4}

■ Solve for L_c

Define the h row and column vectors

In[1136]:=

```
h = Table[If[j == 1, 1, 0], {j, n}]
hr = Reverse[h]
```

Out[1136]=

{1, 0, 0, 0}

Out[1137]=

{0, 0, 0, 1}

In[1138]:=

```
M = Table[h . MatrixPower[φ /. φsoln, i], {i, n}];
M // MatrixForm
```

Out[1139]//MatrixForm=

$$\begin{pmatrix} 1 & T & \frac{T^2}{2} & \frac{T^3}{6} \\ 1 & 2\,T & 2\,T^2 & \frac{4\,T^3}{3} \\ 1 & 3\,T & \frac{9\,T^2}{2} & \frac{9\,T^3}{2} \\ 1 & 4\,T & 8\,T^2 & \frac{32\,T^3}{3} \end{pmatrix}$$

In[1140]:=

```
αφsoln . Inverse[M] . hr;
% // FullSimplify
% // MatrixForm
```

Out[1141]=

$$\left\{1-\beta^4,\ \frac{(-1+\beta)^2\,(11+\beta\,(14+11\,\beta))}{6\,T},\ -\frac{2\,(-1+\beta)^3\,(1+\beta)}{T^2},\ \frac{(-1+\beta)^4}{T^3}\right\}$$

Out[1142]//MatrixForm=

$$\begin{pmatrix} 1-\beta^4 \\ \frac{(-1+\beta)^2\,(11+\beta\,(14+11\,\beta))}{6\,T} \\ -\frac{2\,(-1+\beta)^3\,(1+\beta)}{T^2} \\ \frac{(-1+\beta)^4}{T^3} \end{pmatrix}$$

■ Results of L_c for order of Range[6]

■ 1st order results

$$(\ 1-\beta\ )$$

■ 2nd order results

$$\begin{pmatrix} 1-\beta^2 \\ \frac{(-1+\beta)^2}{T} \end{pmatrix}$$

This verifies the results for the 2nd order from the handout

$$\begin{pmatrix} -(\beta-1)^2 - 2(\beta-1) \\ \frac{(-1+\beta)^2}{T} \end{pmatrix} \text{ // FullSimplify // MatrixForm}$$

$$\begin{pmatrix} 1-\beta^2 \\ \frac{(-1+\beta)^2}{T} \end{pmatrix}$$

■ 3rd order results

$$\begin{pmatrix} 1-\beta^3 \\ \frac{3(-1+\beta)^2(1+\beta)}{2T} \\ -\frac{(-1+\beta)^3}{T^2} \end{pmatrix}$$

■ 4th order results

$$\begin{pmatrix} 1-\beta^4 \\ \frac{(-1+\beta)^2(11+14\beta+11\beta^2)}{6T} \\ -\frac{2(-1+\beta)^3(1+\beta)}{T^2} \\ \frac{(-1+\beta)^4}{T^3} \end{pmatrix}$$

■ 5th order results

$$\begin{pmatrix} 1-\beta^5 \\ \frac{5(-1+\beta)^2(5+7\beta+7\beta^2+5\beta^3)}{12T} \\ -\frac{5(-1+\beta)^3(7+10\beta+7\beta^2)}{12T^2} \\ \frac{5(-1+\beta)^4(1+\beta)}{2T^3} \\ -\frac{(-1+\beta)^5}{T^4} \end{pmatrix}$$

■ 6th order results

$$\begin{pmatrix} 1-\beta^6 \\ \frac{(-1+\beta)^2(137+\beta(202+\beta(222+\beta(202+137\beta))))}{60T} \\ -\frac{5(-1+\beta)^3(1+\beta)(3+\beta(2+3\beta))}{4T^2} \\ \frac{(-1+\beta)^4(17+\beta(26+17\beta))}{4T^3} \\ -\frac{3(-1+\beta)^5(1+\beta)}{T^4} \\ \frac{(-1+\beta)^6}{T^5} \end{pmatrix}$$