Current Discrete Estimator Notes

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Abstract

Derivation notes for the Current Discrete Estimator

References

[1] z.gao, "Current discrete estimator notes," 4/6/2004. [Online]. Available: http://aaron.homeunix.com

EEC 640/740 Dr. Gao 4-6-2004

Gene Franklin: Digital Control of Dynamic Systems 2nd ed

Chap. 6

understand the problem then you see a way around it

$$\begin{cases} \dot{x} = Fx + Gu \\ y = Hx \end{cases} \xRightarrow{ZOH} \begin{cases} x(k+1) = \phi x(k) + \Gamma u(k) \\ y(k) = Hx(k) \end{cases}$$

discretize $\boxed{\phi = e^{FT}}$, $\boxed{\Gamma = \int_0^T e^{F\eta} d\eta\, G}$

cont dig

$F \Rightarrow A$ ϕ Γ

$G \Rightarrow B$

$H \Rightarrow C$

Prediction Estimator:

$$\bar{x}(k+1) = \phi \bar{x}(k) + \Gamma u(k) + L_p [y(k) - H\bar{x}(k)]$$

standard observer in Simulink

Let $\tilde{x} = \bar{x} - x$

$$\tilde{x}(k+1) = (\phi - L_p H)\tilde{x}(k)$$

all estimations come from previous sample

Current time Estimator:

simultaneous diff. equation

old current measurement

$$\hat{x}(k) = \bar{x}(k) + L_c(y(k) - H\bar{x}(k)) \qquad 6.33a$$

$$\bar{x}(k) = \phi \hat{x}(k-1) + \Gamma u(k-1) \qquad 6.33b$$

old data

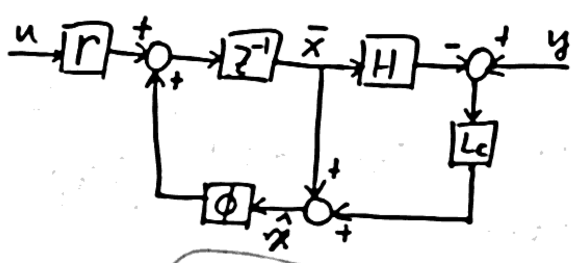
or

$\cancel{\hat{x}(k) = \phi \hat{x}}$ 6.33a → 6.33b

$$\bar{x}(k+1) = \phi \bar{x}(k) + \Gamma u(k) + \phi L_c [y(k) - H\bar{x}(k)]$$

$$\tilde{x}(k+1) = [\phi - \underbrace{\phi L_c H}_{L_p}]\tilde{x}(k) \qquad 6.35$$

$$\Rightarrow \quad L_p = \phi L_c$$



Let $\tilde{x} = \hat{x} - x$ $\Rightarrow$ $\boxed{\tilde{x}(k+1) = [\phi - L_c H \phi]\tilde{x}(k)}$ 6.37

Figure 1: Page 1

Ackermann's Formula

$$\begin{cases} x(k+1) = \phi x(k) + \Gamma u(k) \\ y = Hx(k) \end{cases} \Rightarrow x(k+1) = (\phi - \Gamma K)x(k)$$

$u = -Kx$

Desired roots: $z_i = \beta_1, \beta_2, \cdots \beta_n$ $\quad (z = e^{sT})$

$\cdots$ char. poly: $\alpha_c(z) = (z-\beta_1)(z-\beta_2)\cdots$

$$\boxed{AF: \quad K = [0 \cdots 0\ 1][\Gamma\ \phi\Gamma \cdots \phi^{n-1}\Gamma]^{-1}\alpha_c(\phi)}$$

Example 2.11:

$$\ddot{\theta} = u \Rightarrow \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \theta = [1\ 0]x \end{cases}$$

corresponding ZOH design

Example 2.13

$$\phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Let $z_{1,2} = e^{-\omega_0 T} \Rightarrow \alpha_c(z) = (z - \underbrace{e^{-\omega_0 T}}_{\beta})^2 = z^2 - 2\underbrace{e^{-\omega_0 T}}_{\beta} z + e^{-2\omega_0 T}$

$$\phi^2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix}$$

$$\alpha_c(\phi) = \phi^2 - 2e^{-\omega_0 T}\phi + e^{-2\omega_0 T} I$$

matrix

$$= \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 2e^{-\omega_0 T}\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + e^{-2\omega_0 T} I$$

$$= \begin{bmatrix} 1 - 2e^{-\omega_0 T} + e^{-2\omega_0 T} & 2T(1 - e^{-\omega_0 T}) \\ 0 & 1 - 2e^{-\omega_0 T} + e^{-2\omega_0 T} \end{bmatrix}$$

let $\beta = e^{-\omega_0 T}$

matrix

$$\alpha_c(\phi) = \begin{bmatrix} 1 - 2\beta + \beta^2 & 2T(1-\beta) \\ & 1 - 2\beta + \beta^2 \end{bmatrix} = \begin{bmatrix} (\beta-1)^2 & -2T(\beta-1) \\ 0 & (\beta-1)^2 \end{bmatrix}$$

characteristic of Matrix

Figure 2: Page 2

$$\phi r = \begin{bmatrix} \frac{3}{2}T^2 \\ T \end{bmatrix}$$

$$K = [\,0 \quad 1\,][\,r \quad \phi r\,]^{-1} \alpha_c(\phi)$$

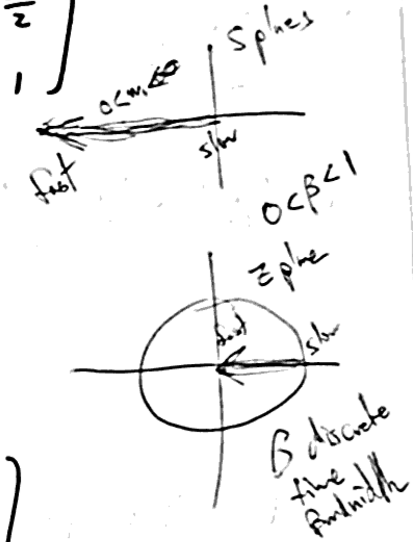
$$[\,r \; \phi r\,] = \begin{bmatrix} T^2/2 & \frac{3}{2}T^2 \\ T & T \end{bmatrix} = T \begin{bmatrix} \frac{T}{2} & \frac{3T}{2} \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{T}{2} & \frac{3T}{2} \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{-T} \begin{bmatrix} 1 & -\frac{3T}{2} \\ -1 & \frac{T}{2} \end{bmatrix}$$

$$[\,r \; \phi r\,]^{-1} = \begin{bmatrix} -\frac{1}{T^2} & +\frac{3}{2T} \\ +\frac{1}{T^2} & -\frac{1}{2T} \end{bmatrix}$$

$$K = \left(\frac{1}{T^2} \quad -\frac{1}{2T}\right) \begin{bmatrix} (\beta-1)^2 & -2T(\beta-1) \\ 0 & (\beta-1)^2 \end{bmatrix}$$

$$= \left[\frac{(\beta-1)^2}{T^2} \quad \left(-\frac{2(\beta-1)}{T} - \frac{(\beta-1)^2}{2T} \right) \right]$$



$$\boxed{\beta = e^{-\omega_0 T}}$$

mapping

- $\{\omega_0 : 0 \le \omega_0 < \infty\}$ maps to $\{\beta : 0 < \beta \le 1\}$

stability

- β is the bandwidth of discrete-time closed-loop system
- Nyquist freq: $2\omega_0 < \boxed{2\pi/T} \Rightarrow \omega_0 T < \pi \Rightarrow \beta > .04$
- $\frac{2\pi}{T} > 10\,\omega_0$: $\omega_0 T < .2\pi \Rightarrow \beta > .53$

Figure 3: Page 3

2aa

in observer this is what we have.

assign eigenvalues of $\boxed{\phi - L_c H \phi.}$ but AF is

for $\phi - \Gamma K$ $\Rightarrow$ assign eigenvalues of $\phi^T - \phi^T H^T L_c^T$

dual to sf to obs form

$\Rightarrow$ $\phi^T \to \phi$, $\phi^T H^T \to \Gamma$, & $L_c^T \to K$ in AF.

$$L_c^T = K = [0 \cdots 0\,1][\phi^T H^T \;\; \phi^{2T} H^T \cdots \phi^n H^T]^{-1} \alpha_e(\phi^T).$$

Ackermann's formula for state observer design

$$\boxed{\Rightarrow \; L_c = \alpha_e(\phi) \begin{bmatrix} H\phi \\ H\phi^2 \\ \vdots \\ H\phi^n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

for example 2.11, 2.13

for double integrator

$$\alpha_e(\phi) = \begin{bmatrix} (\beta-1)^2 & -2T(\beta-1) \\ 0 & (\beta-1)^2 \end{bmatrix}, \qquad H\phi = [1 \;\; T], \quad H\phi^2 = [1 \;\; 2T]$$

$$M = \begin{bmatrix} H\phi \\ H\phi^2 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 1 & 2T \end{bmatrix} \qquad M^{-1} = \frac{1}{T} \begin{bmatrix} 2T & -T \\ 1 & 1 \end{bmatrix},$$

$$L_c = \begin{bmatrix} (\beta-1)^2 & -2T(\beta-1) \\ 0 & (\beta-1)^2 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1}{T} \end{bmatrix} = \begin{bmatrix} -(\beta-1)^2 - 2(\beta-1) \\ \frac{(\beta-1)^2}{T} \end{bmatrix}$$

Figure 4: Page 4

Stability?

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Delay and Phase lag due to sampling:

Delay of T: e^{-sT} pure delay

phase lag: $\angle -\omega T$ r/s

at $\omega = \frac{1}{2}\frac{2\pi}{T} = \frac{\pi}{T}$, $\angle -\omega T = -\pi \Rightarrow$ unstable!

at $\omega = \frac{1}{10}\frac{2\pi}{T}$: $\angle -\omega T = -\frac{\pi}{5} \Rightarrow -35^\circ \Rightarrow$ dangerous!

- Average delay of Sample and Hold : $\frac{T}{2}$
- Delay in the Predicative Estimator : at least T
- . . . . Current . . . . : $? < t_d < T$
- Total delay in a digital implementation of an observer based controller $= \underbrace{\frac{T}{2}}_{\text{Sampling}} + \underbrace{T}_{\text{observer}} + \underbrace{\frac{T}{2}}_{\text{D/A}} \approx 2T$

let $\omega_s = 10\omega_c \Rightarrow -70^\circ$ phase lag !!

This is why you see oscillations in digital control as you crank up the bandwidth.

Figure 5: Page 5

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Discrete ESO

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (1) \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = [1 \ 0 \ 0]$$

$$\Downarrow \quad f_s = \frac{1}{T}$$

for observer design

$$\begin{cases} x(k+1) = \phi x(k) + \Gamma u(k) \\ y(k) = C x(k) \end{cases} \qquad (2)$$

PDESO (predicative, discrete) (Luenberger) (Simulink.

$$\bar{x}(k+1) = \phi \bar{x}(k) + \Gamma u(k) + L_p[y(k) - C\bar{x}(k)] \qquad (3)$$

$$\tilde{x} = \bar{x} - x$$

$$\tilde{x}(k+1) = \phi \tilde{x} + L_p C(x(k) - \bar{x}) = (\phi - L_p C)\tilde{x}(k) \quad (4$$

CDESO (current, discrete)

$$\hat{x}(k) = \bar{x}(k) + L_c(y(k) - C\bar{x}(k)) = (I - L_c C)\bar{x} + L_c y(k) \quad (5$$

$$\bar{x}(k) = \phi \hat{x}(k-1) + \Gamma u(k-1)$$

$$\Rightarrow \hat{x}(k) = (I - L_c C)\phi \hat{x}(k-1) + (I - L_c C)\Gamma u(k-1) + L_c y(k)$$

new ESO eq.

$$= \underbrace{(I - L_c C)\phi}_{\Phi_c} \hat{x}(k-1) + \Big[\underbrace{(I - L_c C)\Gamma \quad L_c}_{\Gamma_c}\Big] \begin{bmatrix} u(k-1) \\ y(k) \end{bmatrix} \qquad (6)$$

CDESO $\left\{ \phi_c, \Gamma_c, I_3, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

- not directly implementable using the discrete state space model in simulink.
- use s-function

Figure 6: Page 6

Determine ϕ, Γ from A, B, T (F.P.W 2nd pp.52)

$$\phi = e^{AT} = I + AT + \frac{A^2T^2}{2!} + \frac{A^3T^3}{3!} + \cdots$$

$$= I + AT\psi \qquad (7)$$

$$\psi = I + \frac{AT}{2!} + \frac{A^2T^2}{3!} + \cdots \quad \sum A$$

$$\Gamma = \int_0^T e^{A\eta} d\eta \, B = \sum_{k=0}^{\infty} \frac{A^k T^{k+1}}{(k+1)!} B = \psi T B \qquad (8)$$

$$\psi \approx I + \frac{AT}{2}\left(I + \frac{AT}{3}\left(I + \frac{AT}{4}\left(\cdots \frac{AT}{N-1}\left(I + \frac{AT}{N}\right)\right)\cdots\right)\right)$$

(better numerical condition)

For $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^3 = [0]_{3\times 3}$$

$$\phi = I + AT + \frac{A^2T^2}{2} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}$$

$$\psi = I + \frac{AT}{2} + \frac{A^2T^2}{6} = \begin{bmatrix} 1 & \frac{T}{2} & T^2/6 \\ 0 & 1 & \frac{T}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma = \psi T B = \begin{bmatrix} T^2/2 \\ T \\ 0 \end{bmatrix}$$

Figure 7: Page 7

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Let observer poles be $-\omega_0$ in s and $e^{-\omega_0 T}$ in z

$$\Rightarrow \alpha_e(z) = \left(z - e^{-\omega_0 T}\right)^3 = z^3 - 3\underbrace{e^{-\omega_0 T}} z^2 + 3e^{-2\omega_0 T} z - e^{-3\omega_0 T}$$

$$(z-\beta)^3 = z^3 - 3\beta z^2 + 3\beta^2 z - \beta^3$$

$$L_c = \alpha_e(\phi) \begin{bmatrix} C\phi \\ C\phi^2 \\ C\phi^3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(use mathematica)} \qquad (9)$$

(3×3) (3×3) (3×1)

$$L_C \cdot \overset{3\times 1}{\begin{bmatrix} \; \end{bmatrix}}$$

Figure 8: Page 8