I. Concepts and Tools

- Mathematics for Dynamic Systems
  - Differential Equation
  - Transfer Function
  - State Space
- Time Response
  - Transient
  - Steady State
- Frequency Response
  - Bode and Nyquist Plots
  - Stability and Stability Margins
- Extensions to Digital Control

A Differential Equation of Motion

- Newton's Law:

$$\ddot{y}(t) = f(t, y(t), \dot{y}(t), w(t)) + bu(t)$$

- A Linear Approximation:

$$\ddot{y}(t) = -\frac{\mu}{J}\dot{y}(t) + w(t) + \frac{K_T}{J}u(t)$$

Laplace Transform and Transfer Function

$$Y(s) = \int_0^\infty y(t)e^{-st}dt$$

$$\ddot{y}(t) = -\frac{\mu}{J}\dot{y}(t) + \frac{K_T}{J}u(t)$$

$$\Downarrow$$

$$s^2Y(s) = (-\frac{\mu}{J})sY(s) + \frac{K_T}{J}U(s)$$

$$\Downarrow$$

$$\frac{Y(s)}{U(s)} = G_p(s) = \frac{K_T}{s(Js+\mu)}$$

State Space Description

$$\ddot{y}(t) = -\frac{\mu}{J}\dot{y}(t) + \frac{K_T}{J}u(t)$$

$$\Downarrow$$

$$x_1 = y, x_2 = \dot{y}$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\mu}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_T}{J} \end{bmatrix} u$$

$$y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Let\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\mu}{J} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{K_T}{J} \end{bmatrix}$$

$$C = [1 \;\; 0],\ \mathrm{D=0}$$

From State Space to Transfer Function

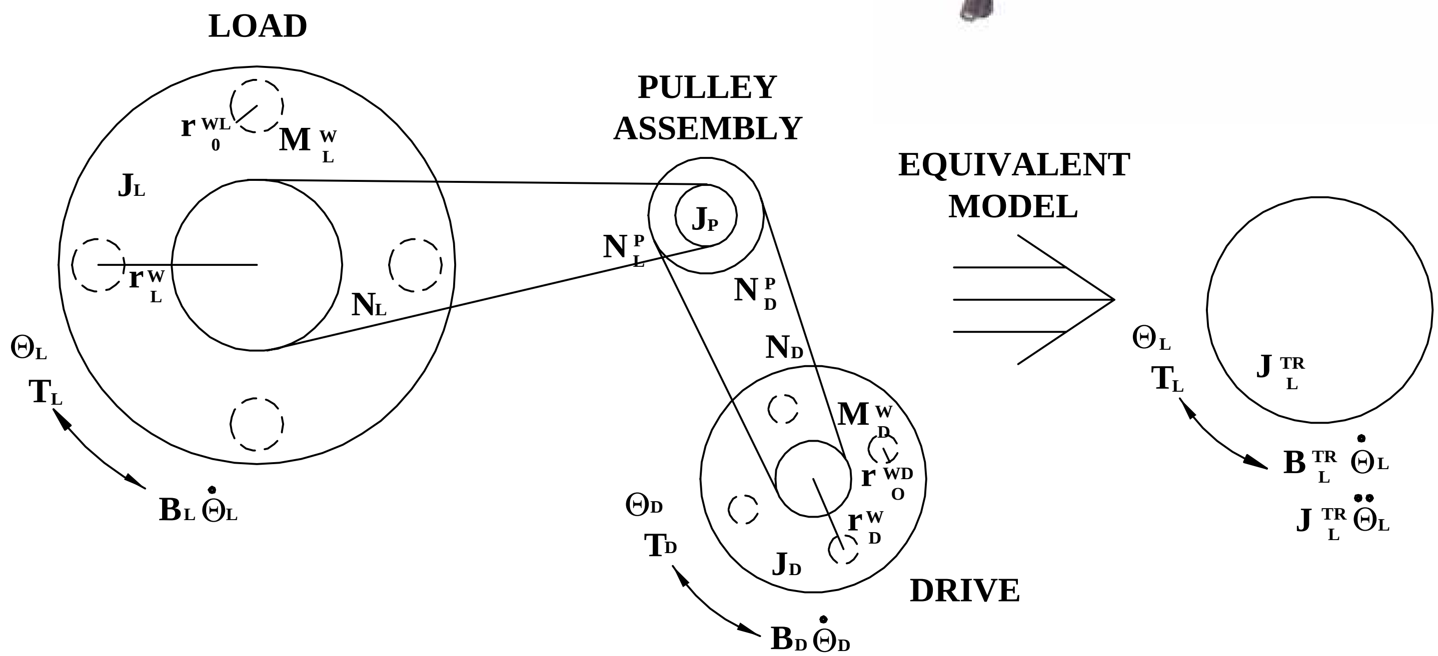
$$G_p(s) = C(sI - A)^{-1}B + D$$

$$= \frac{K_T}{s(Js + \mu)}$$

Linear System Concepts

- States form a linear vector space
- Controllable Subspace and Controllability
- Observable Subspace and Observability
- The Linear Time Invariance (LTI) Assumptions
- Stability
  - Lyapunov Stability (for linear or nonlinear systems)
  - LTI System Stability: poles/eigenvalues in RHP

A Motion Control Problem



From Differential Eq. To Transfer Function

$$T_L = J_T \cdot \ddot{\Theta} + B_T \cdot \dot{\Theta}$$

$$T_L = T_D \cdot gr_T$$

$$TF = \frac{\Theta_L}{T_D} = \frac{gr_T}{s\left(J_T \cdot s + B_T\right)}$$

$$T_D = v_{CS} \cdot K_S \cdot K_A \cdot K_T$$

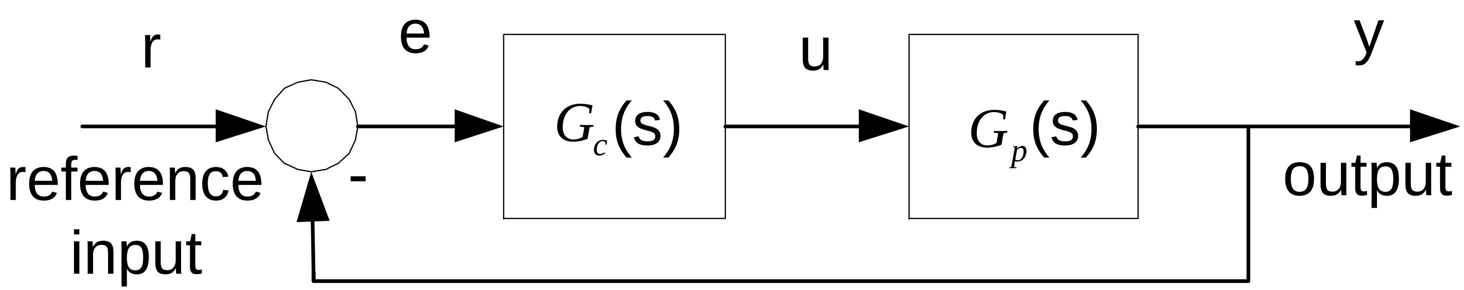
$$G_p(s) = \frac{\Theta_L}{v_{CS}} = \frac{K_S \cdot K_A \cdot K_T \cdot gr_T}{s\left(J_T \cdot s + B_T\right)}$$

Transfer Function model of the motion plant

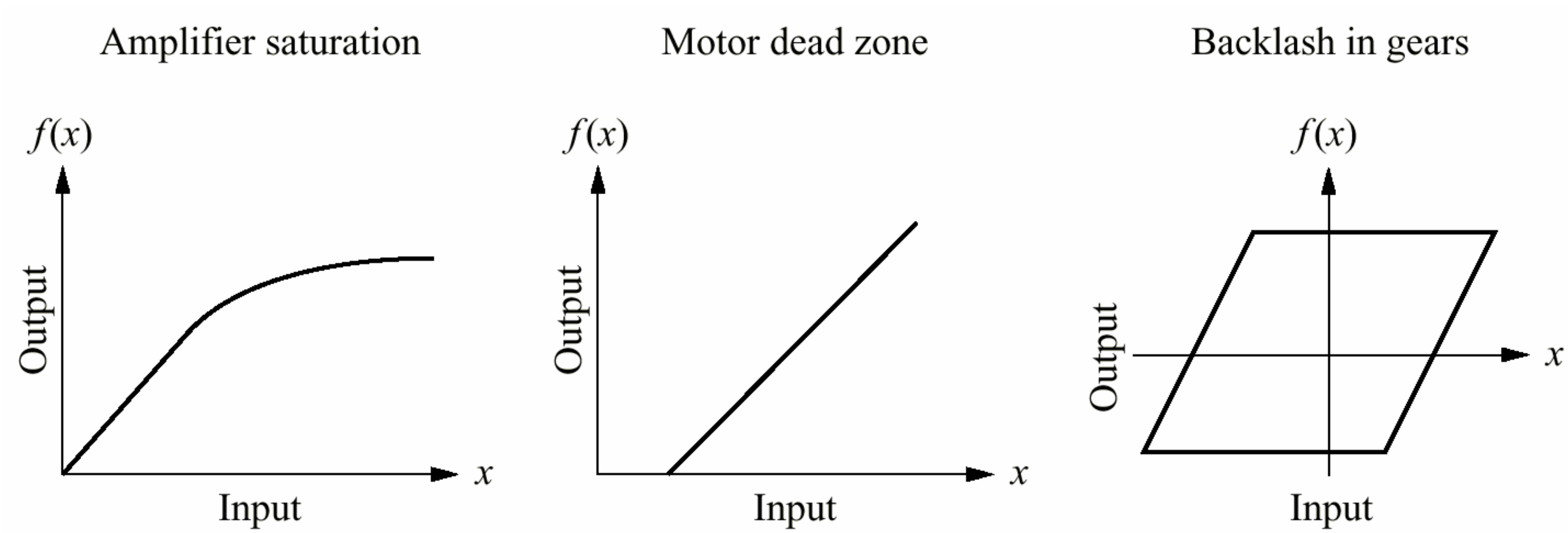
$$G_p(s) = \frac{K_S \cdot K_A \cdot K_T \cdot gr_T}{s\left(J_T \cdot s + B_T\right)}$$

$$G_p(s) = \frac{K}{s\left(Js + \alpha\right)}$$

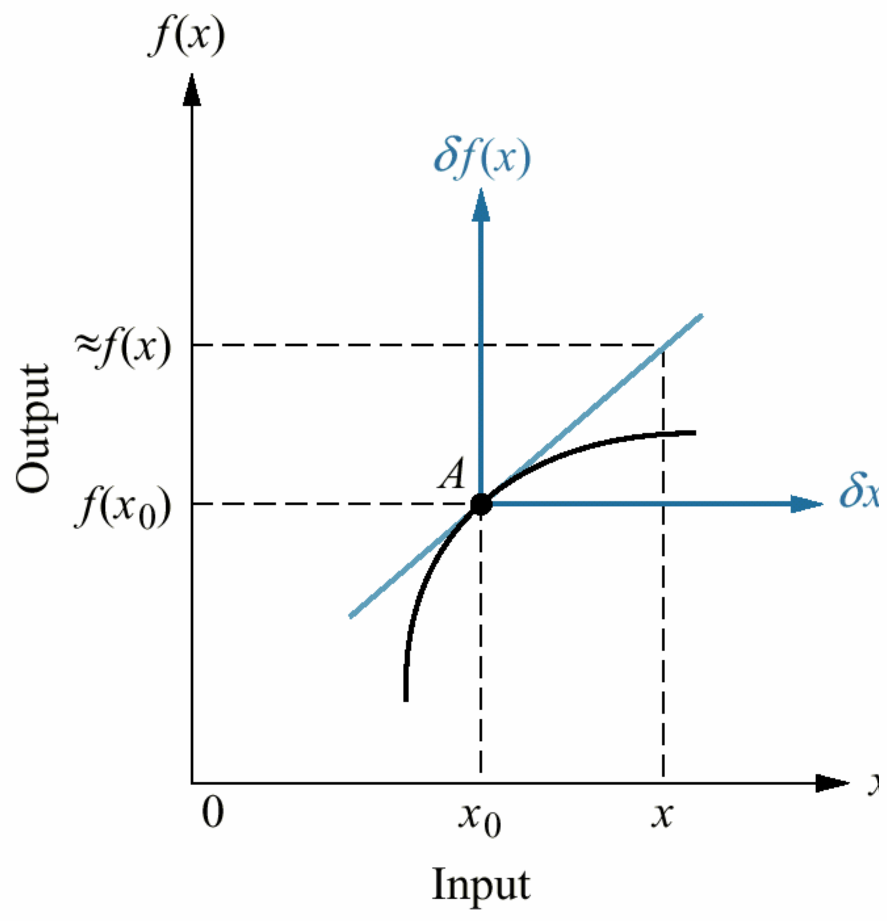
Unity Feedback Control System

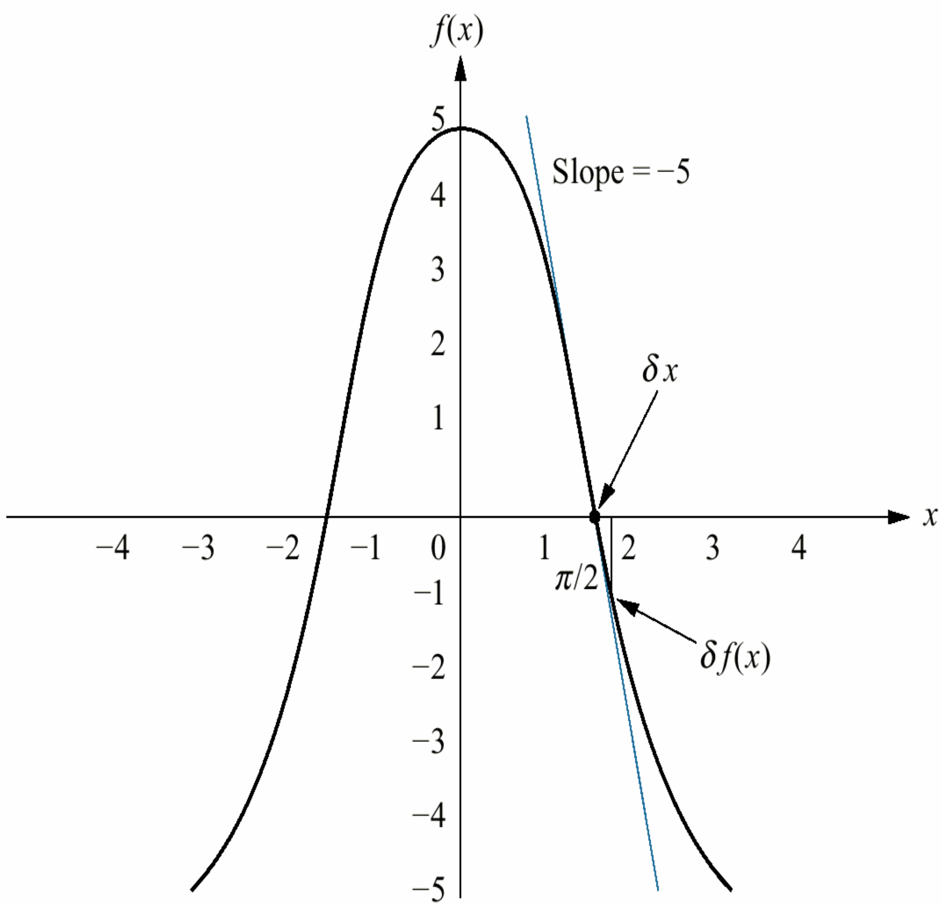


Common Nonlinearities

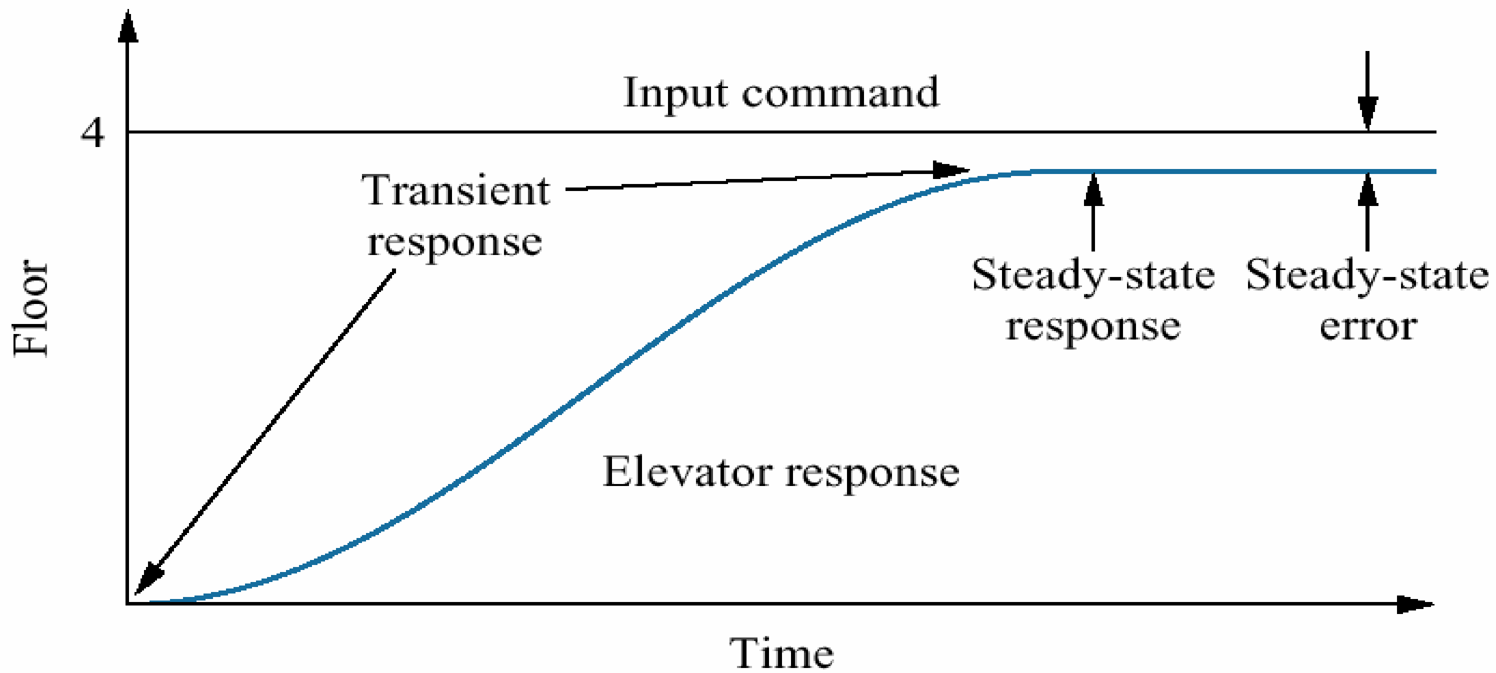


Linearization



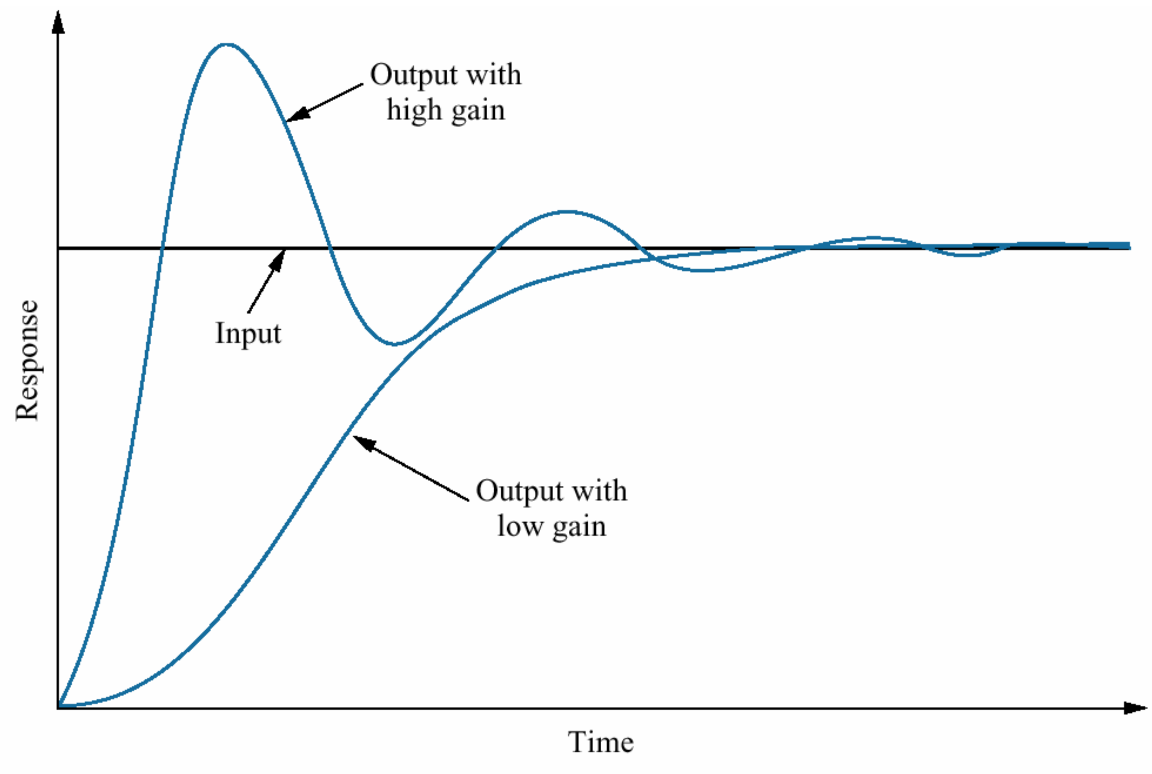


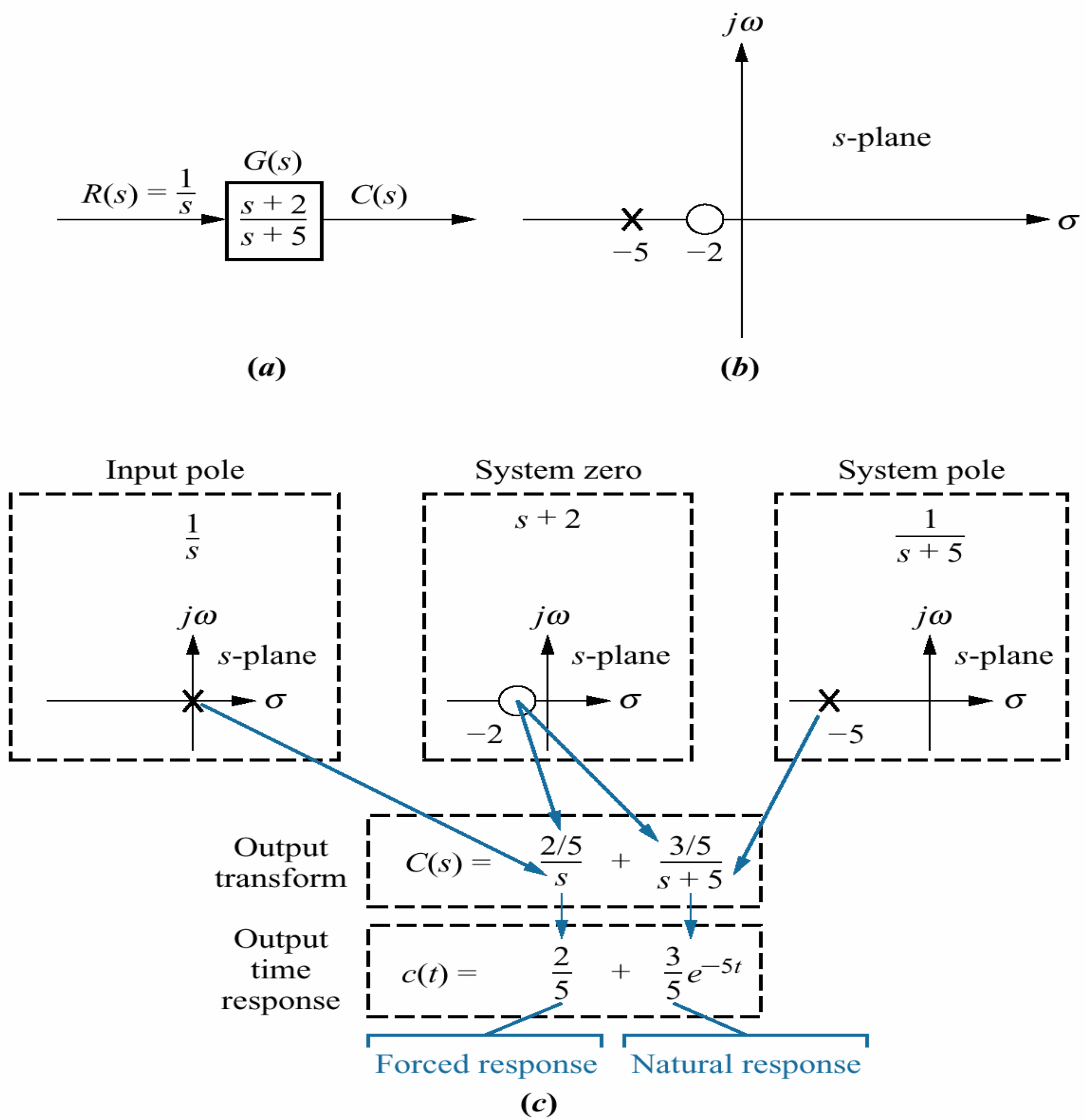
Time Response



Open Loop Transient Response

- How parameters of transfer functions affect output
- Terminologies for 1st and 2nd order systems

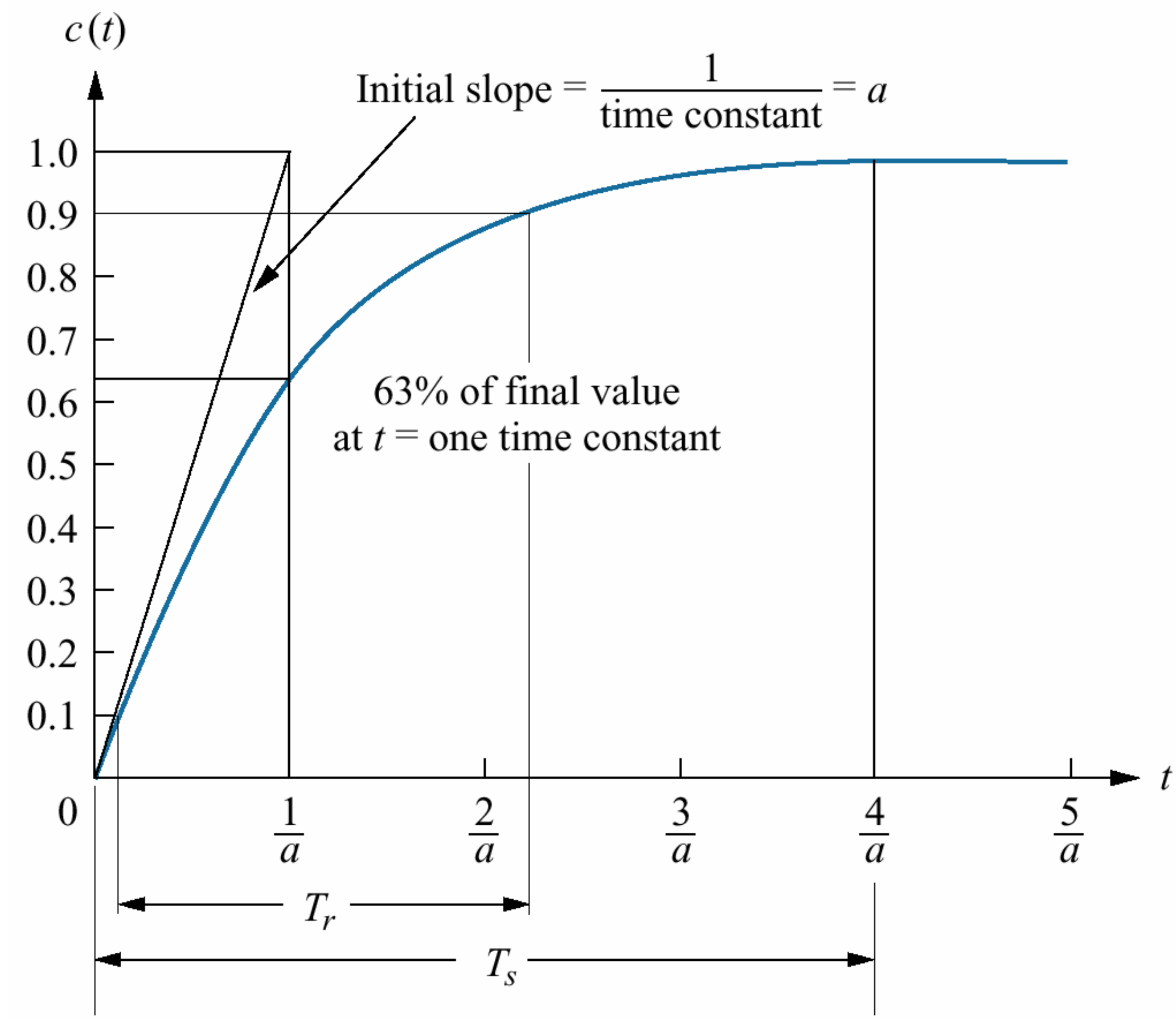




G(s)
R(s) = 1/s
(s + 2)/(s + 5)
C(s)
(a)
jω
s-plane
σ
−5
−2
(b)
Input pole
1/s
jω
s-plane
σ
System zero
s + 2
jω
s-plane
σ
−2
System pole
1/(s + 5)
jω
s-plane
σ
−5
Output transform
C(s) = 2/5 / s + 3/5 / (s + 5)
Output time response
c(t) = 2/5 + 3/5 e−5t
Forced response
Natural response
(c)

First Order Transfer Function

$$G(s) = \frac{a}{(s+a)} = \frac{1}{(\tau s + 1)}$$



Pure 2nd Order Transfer Functions

$$G(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\xi\left(\frac{s}{\omega_n}\right) + 1}$$

ξ : damping ratio, ω_n : natural frequency

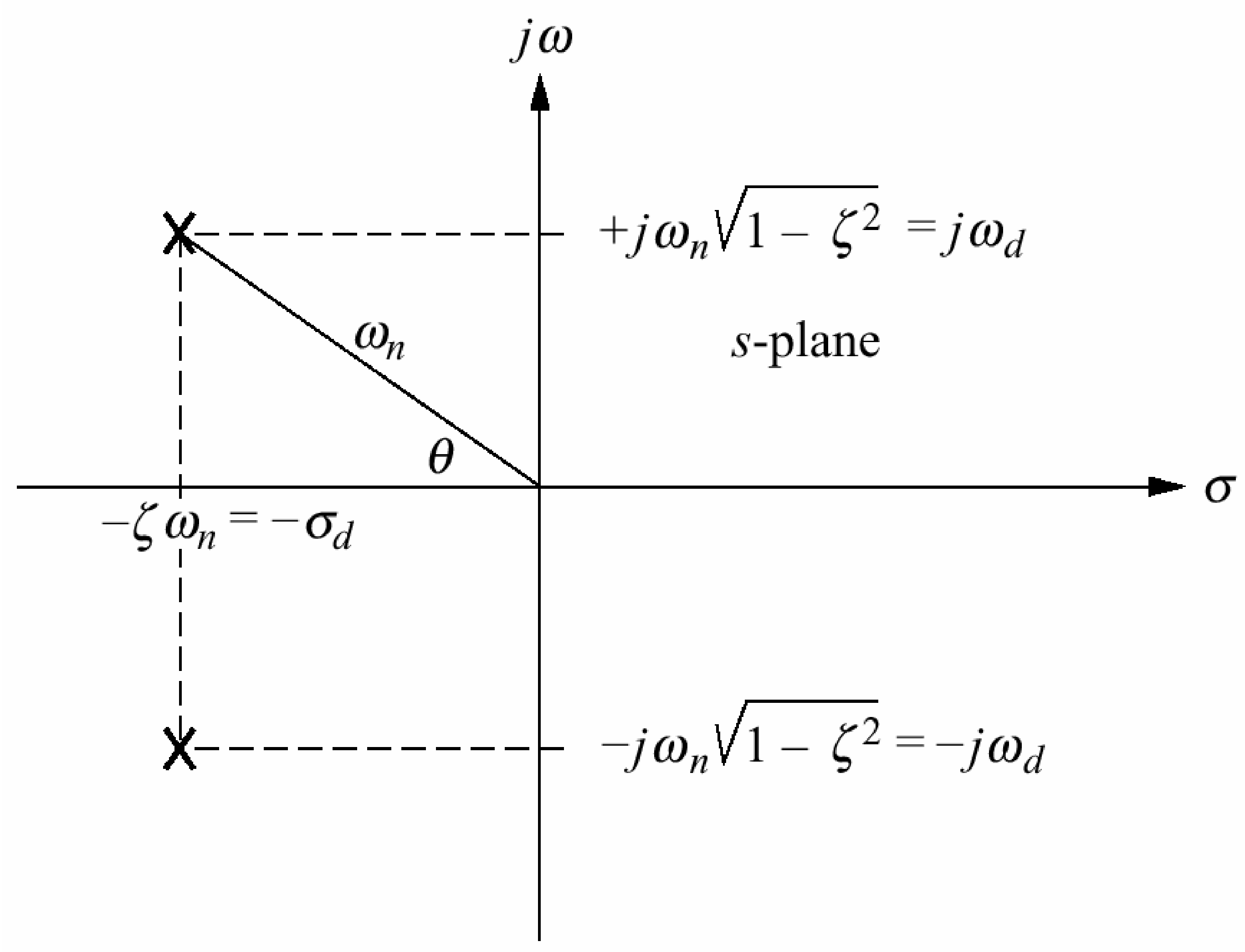
$\xi > 1$: overdamped

$\xi < 1$: underdamped

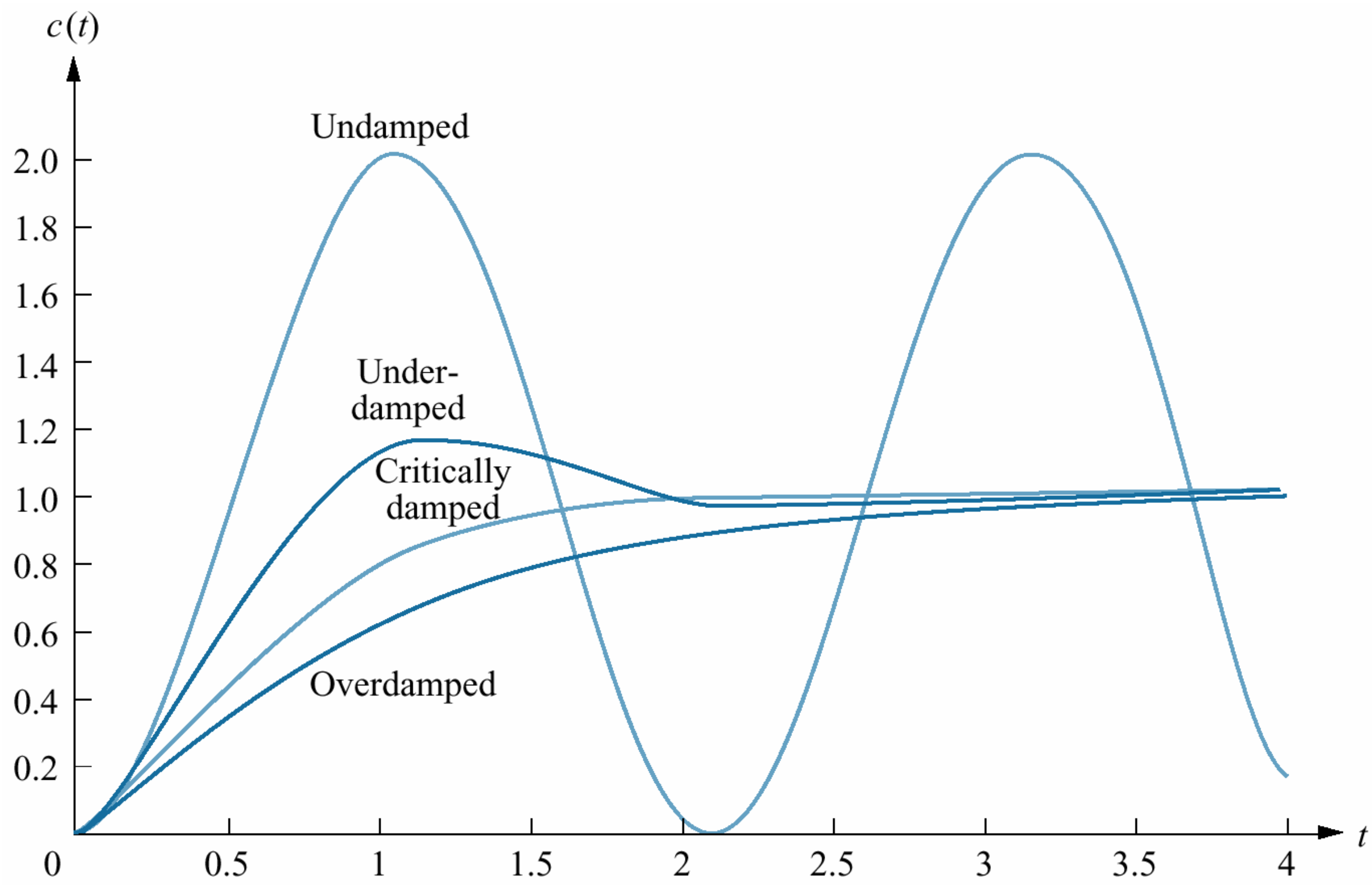
$\xi = 1$: critically damped

$\xi = 0$: undamped

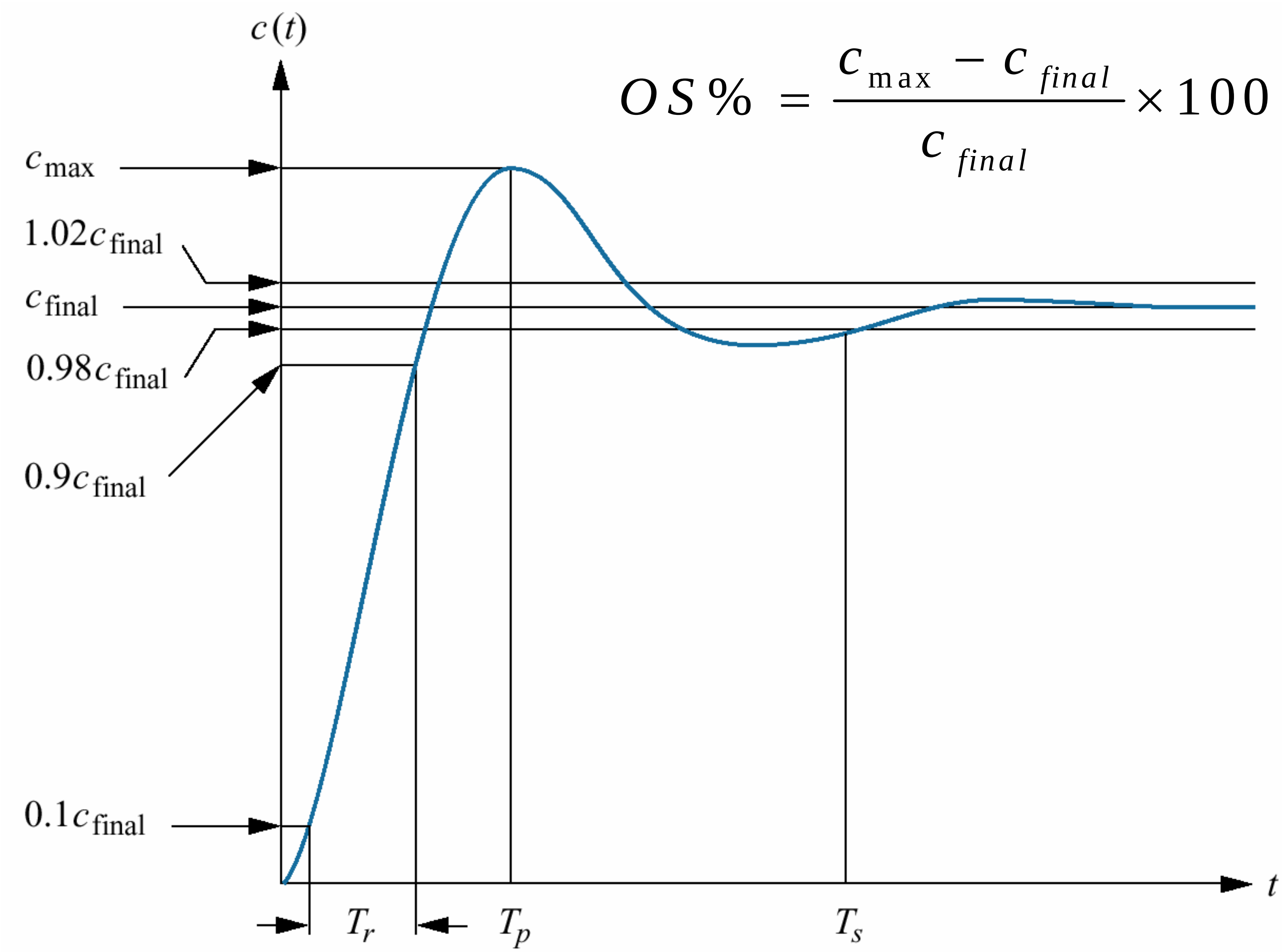
Pure 2nd Order Transfer Functions



Pure 2nd Order Transfer Functions



Terminologies

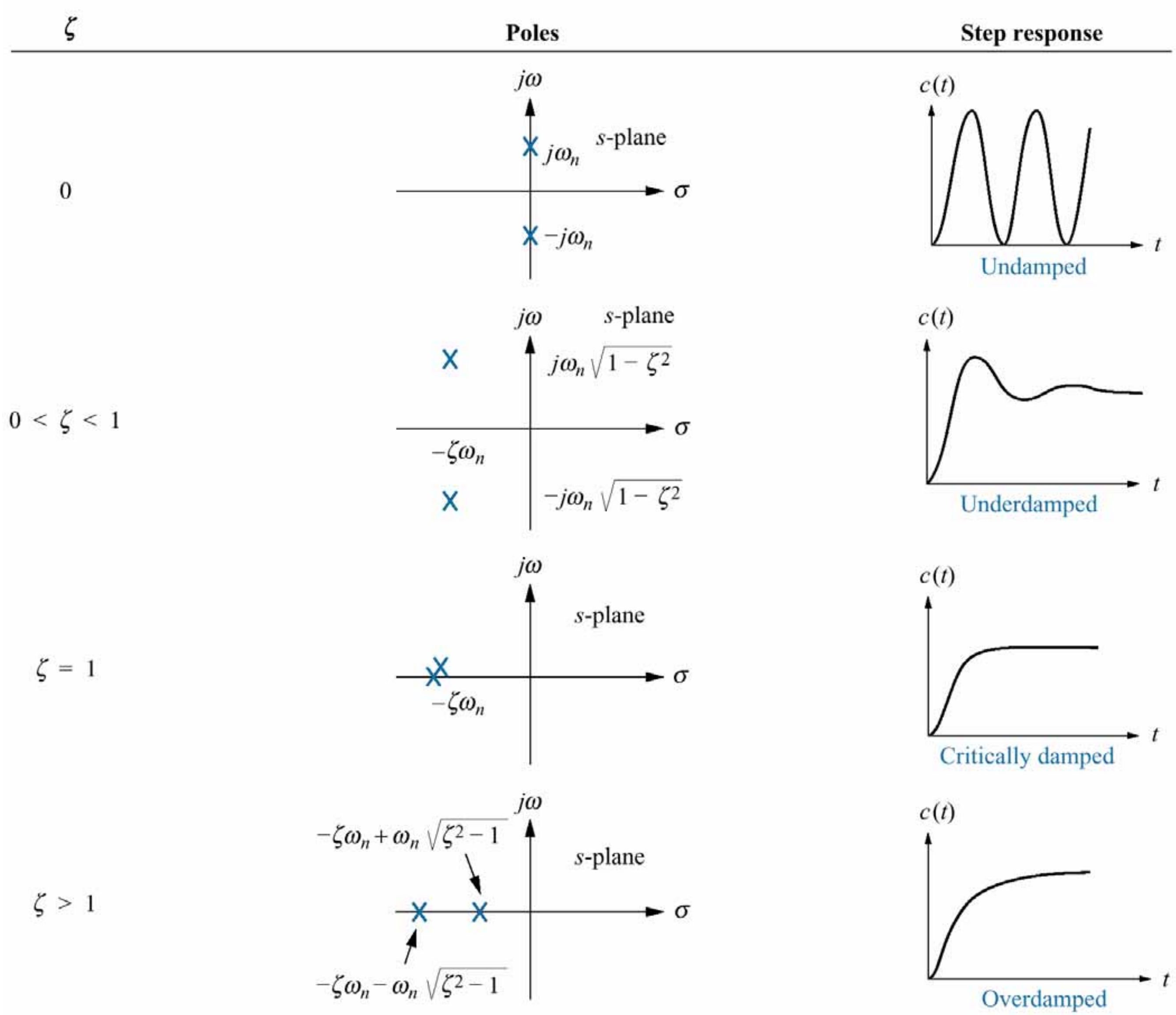


Calculations

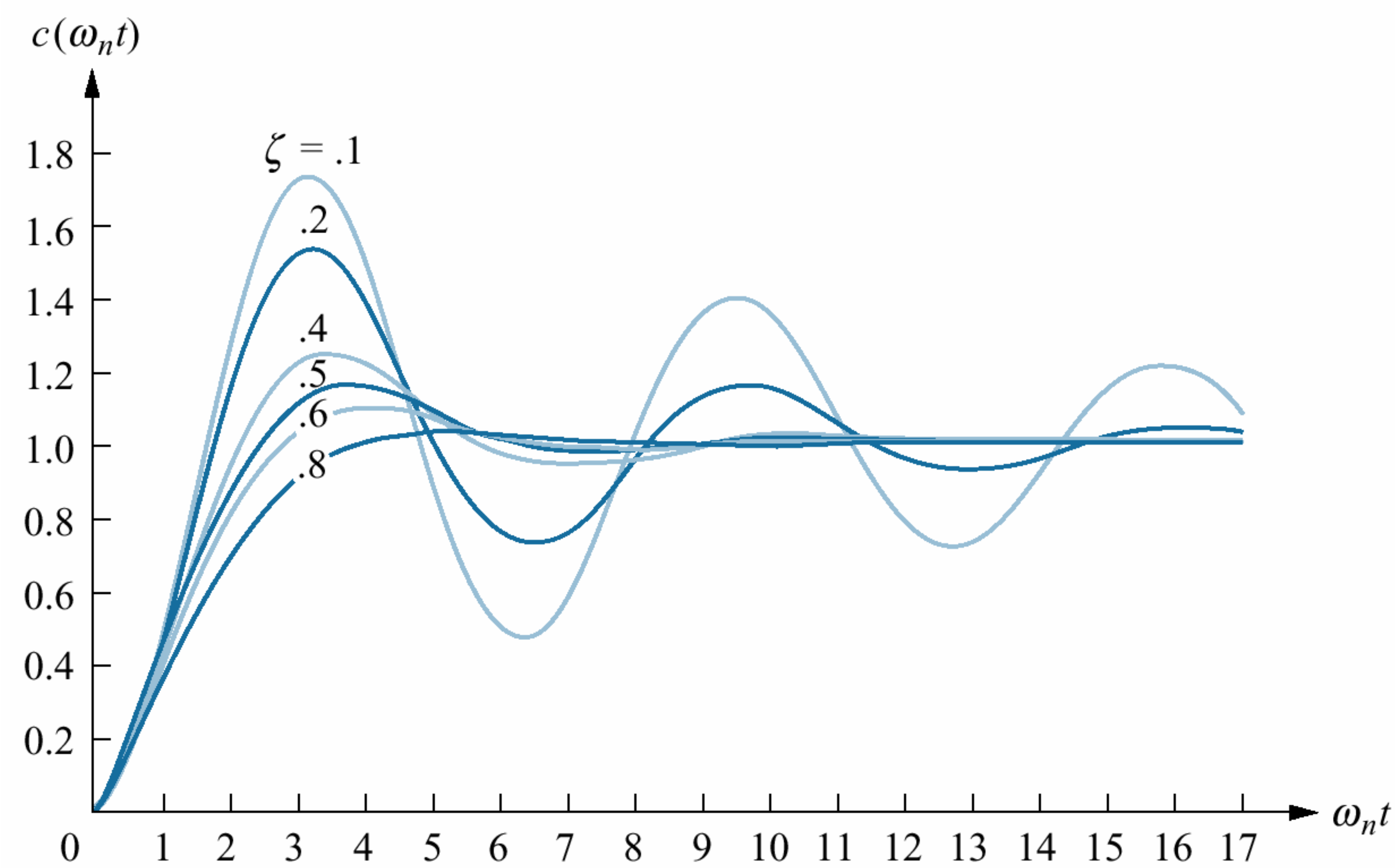
$$OS\% = e^{-(\xi\pi/\sqrt{1-\xi^2})} \times 100$$

$$T_s = \frac{4}{\xi\omega_n}$$

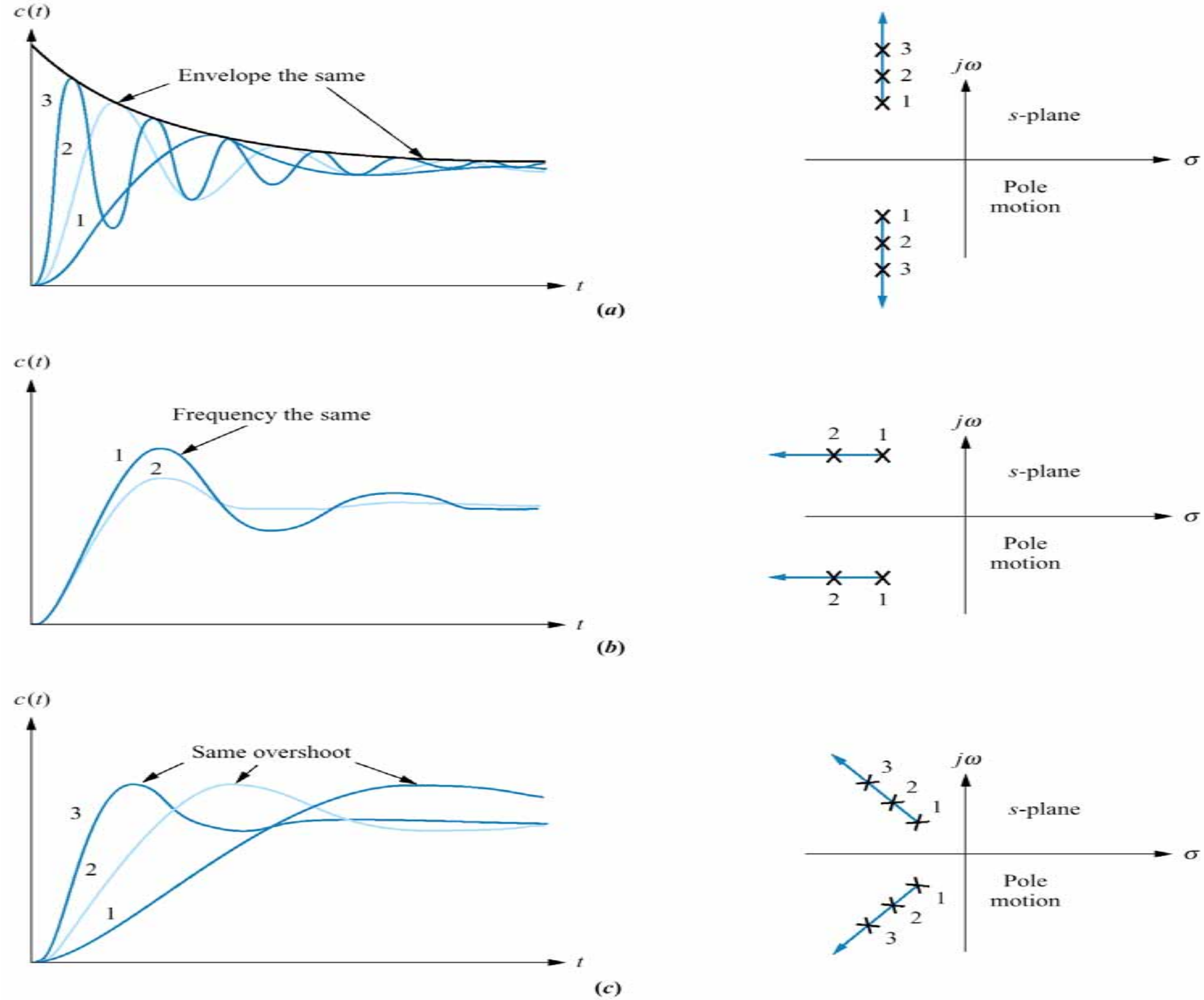
Pure 2nd Order Transfer Functions



Pure 2nd Order Transfer Functions



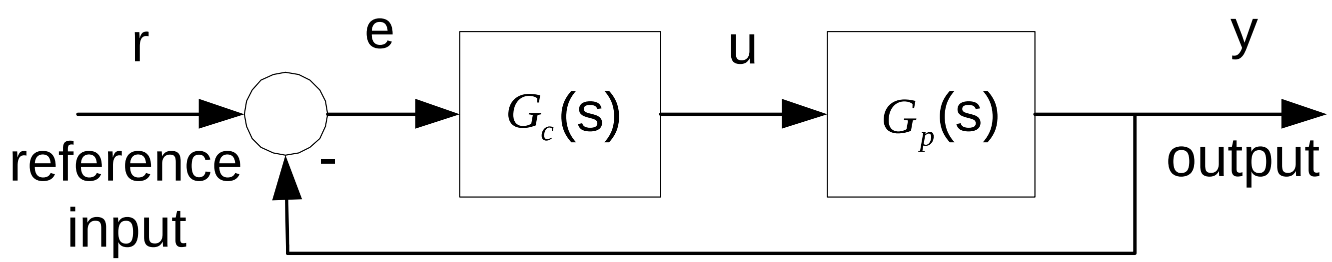
Pure 2nd Order Transfer Functions



Steady State Response

Steady state response is determined by the dc gain: G(0)

Steady State Error



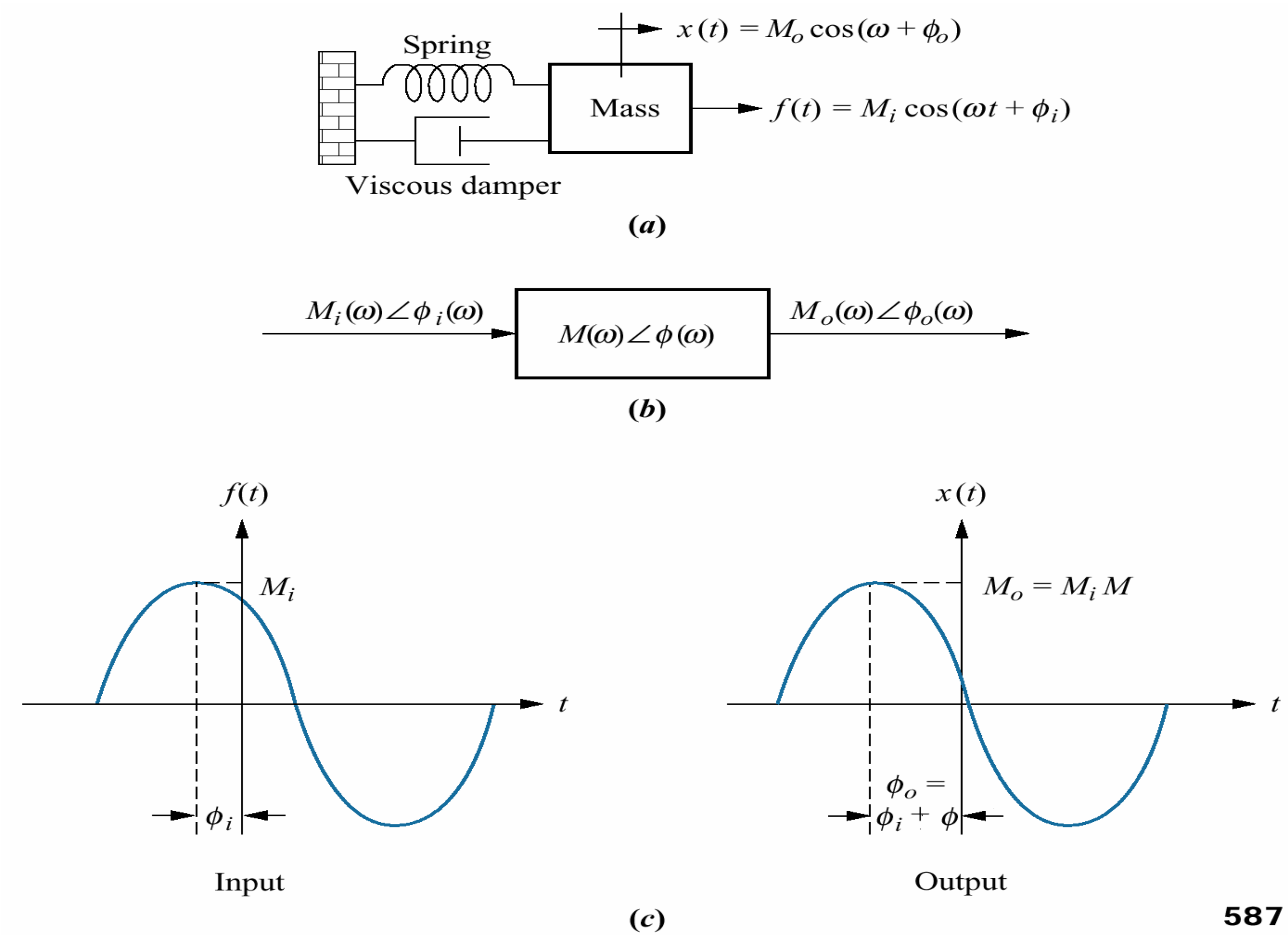
$$e_{ss} = e(t)\big|_{t\to\infty} = sE(s)\big|_{s\to 0}$$

$$E(s) = \frac{R(s)}{1 + G_c(s)G_p(s)}$$

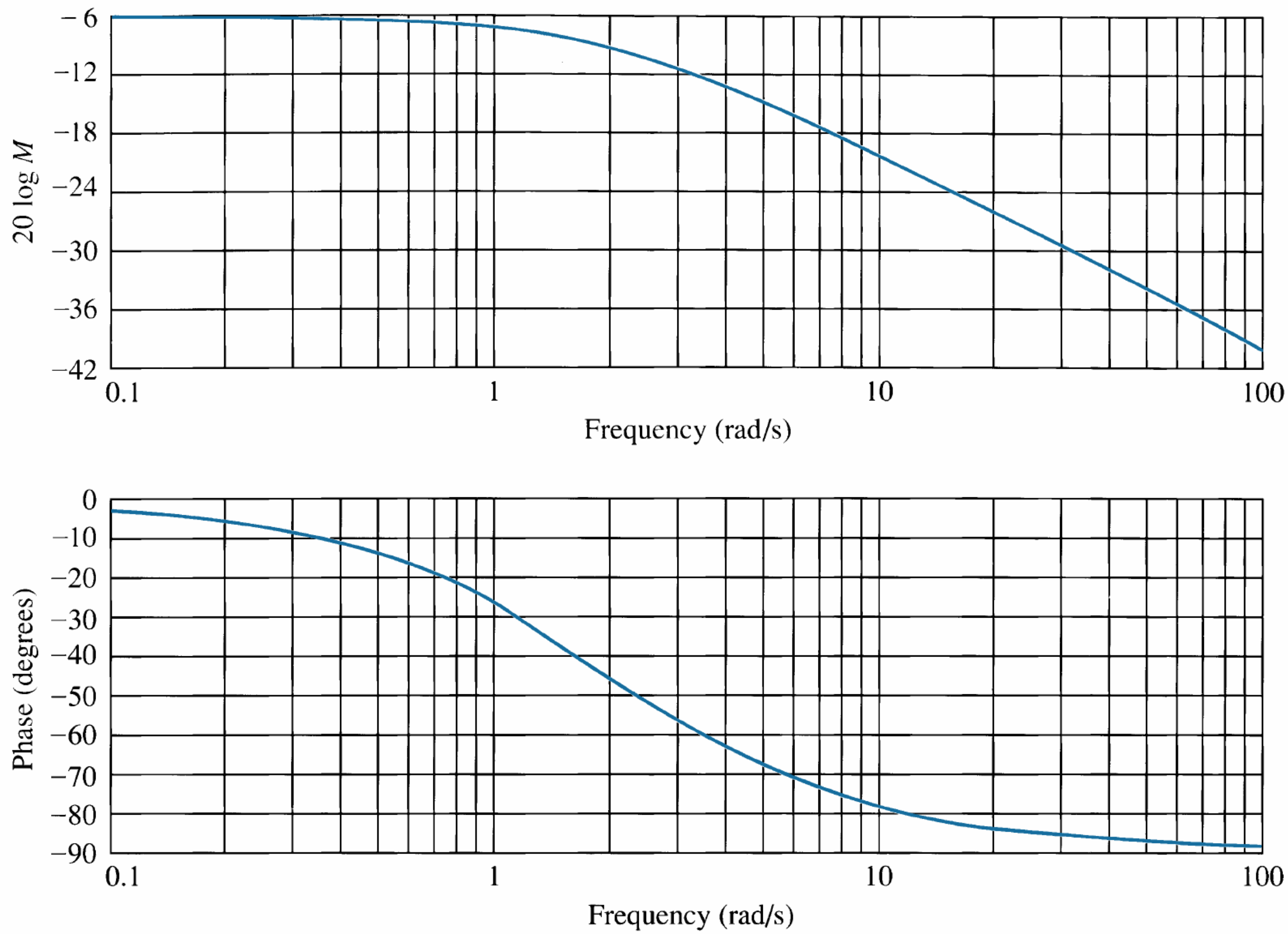
Frequency Response:
The MOST useful concept in control theory

- Performance Measures
  - Bandwidth
  - Disturbance Rejection
  - Noise Sensitivity
- Stability
  - Yes or No?
  - Stability Margins (closeness to instability)
  - Robustness (generalized stability margins)

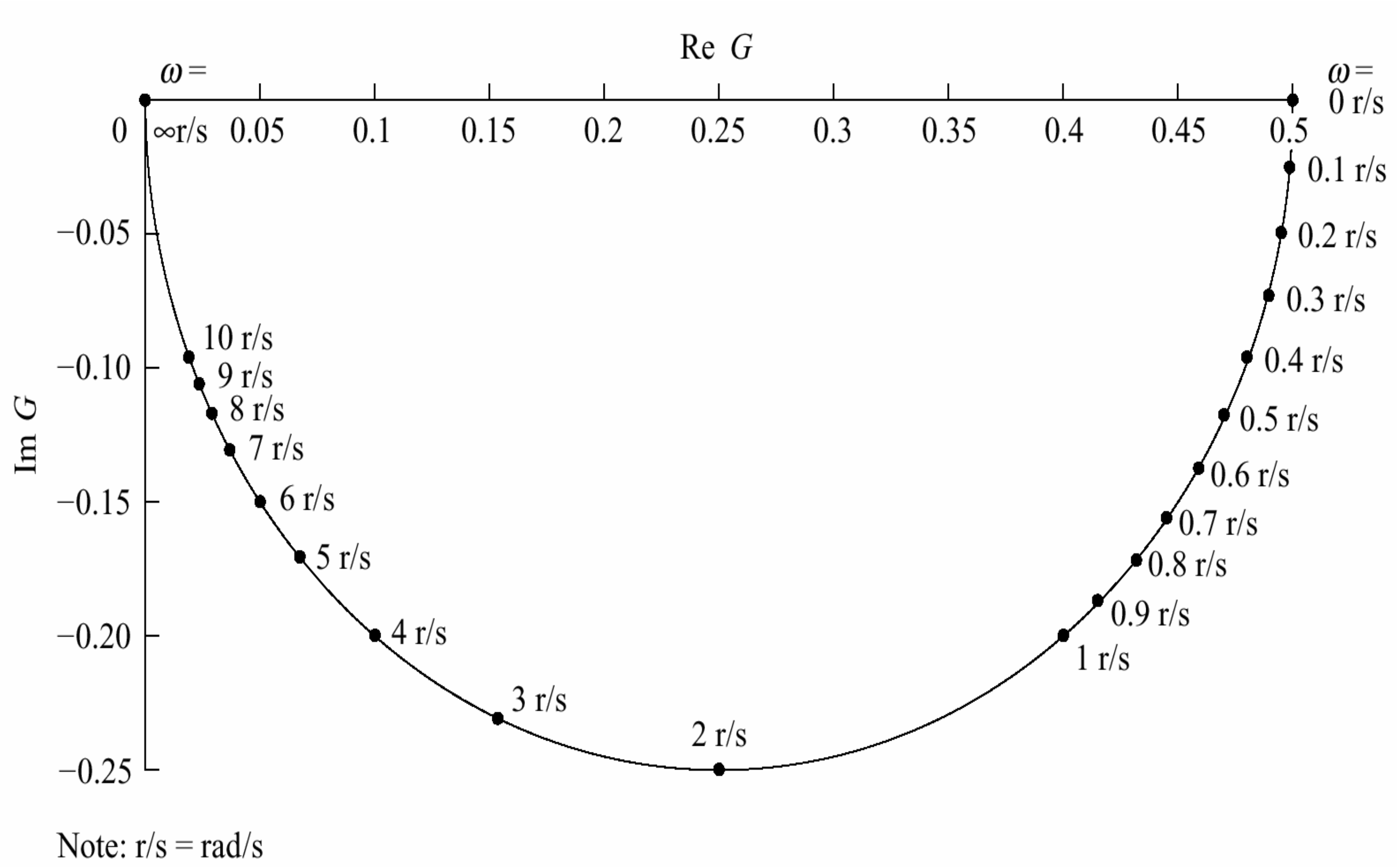
Frequency Response



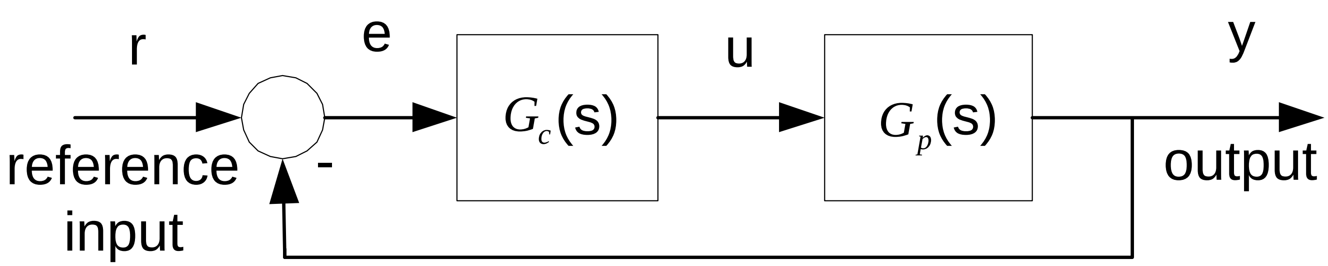
Bode Plot (Magnitude and Phase vs. Frequency) G(s) =1/(s +2)



Polar Plot: imaginary part vs. real part of G(jω)
G(s) =1/(s +2)

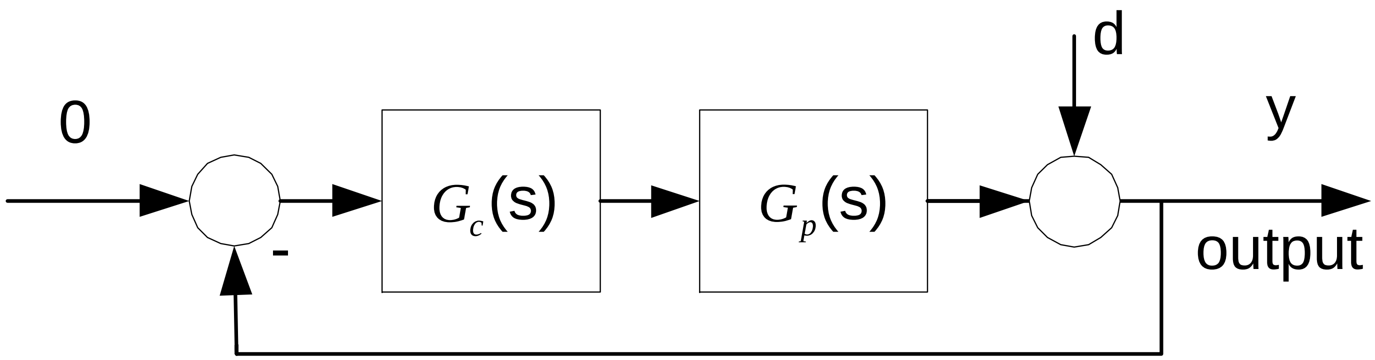


Bandwidth of Feedback Control



- -3dB Frequency of CLTF $\dfrac{Y(j\omega)}{R(j\omega)} = \dfrac{G_c(j\omega)G_p(j\omega)}{1+G_c(j\omega)G_p(j\omega)}$
- 0 dB Crossing Frequency (ω_c) of $G_c(j\omega)G_p(j\omega)$
- Defines how fast y follows r

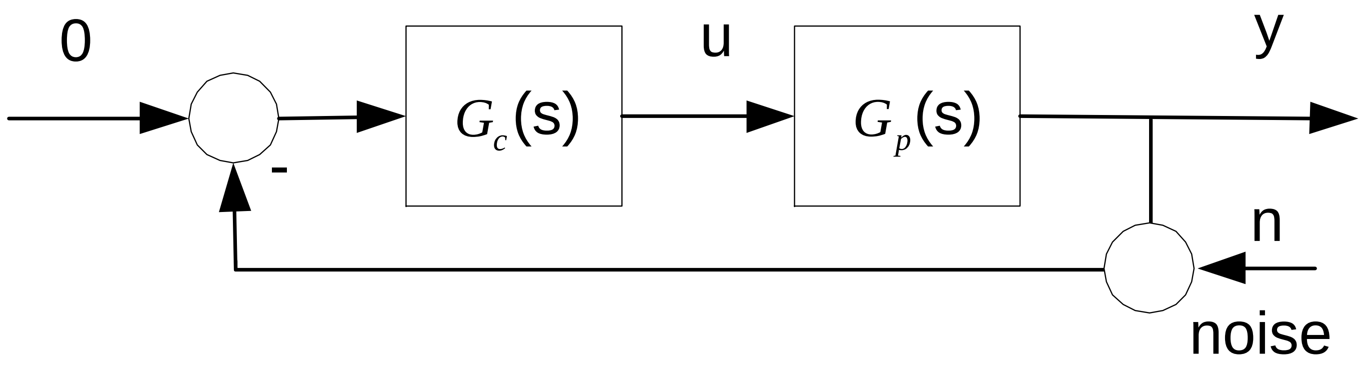
Disturbance Rejection



$$\frac{Y(j\omega)}{D(j\omega)} = \frac{1}{1 + G_c(j\omega)G_p(j\omega)}$$

measures disturbance rejection quality

Noise Sensitivity

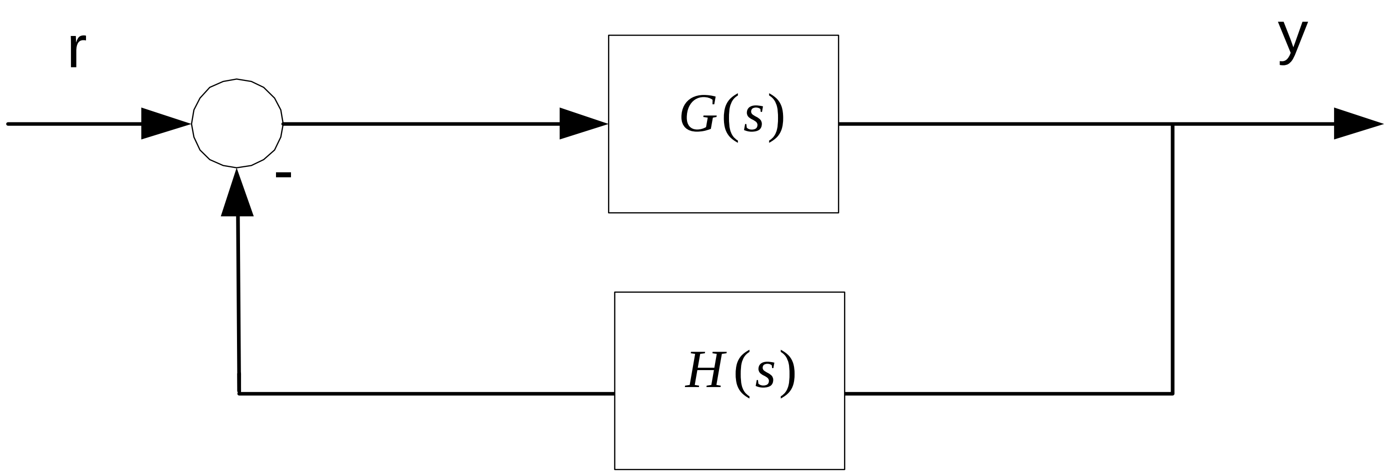


$$\frac{U(j\omega)}{N(j\omega)} = \frac{-G_c(j\omega)}{1+G_c(j\omega)G_p(j\omega)}$$

$$\approx -G_c(j\omega) \text{ at high frequency}$$

Nyquist Plot

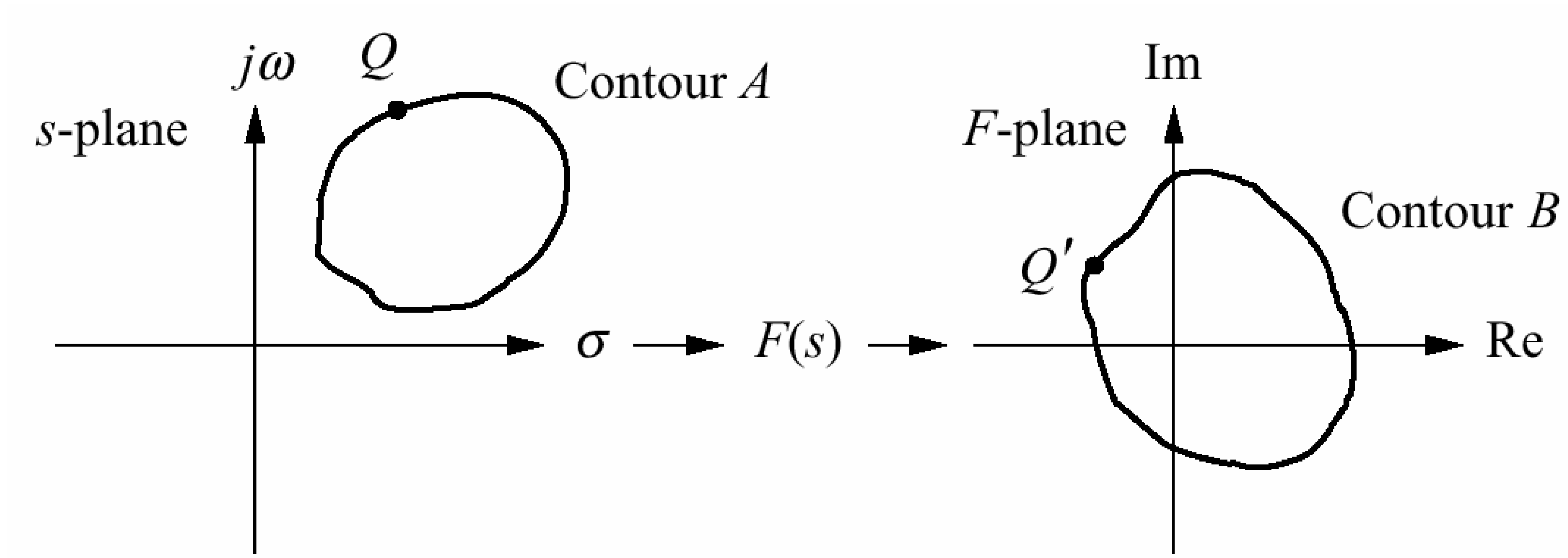
Using G (jω) to determine the stability of



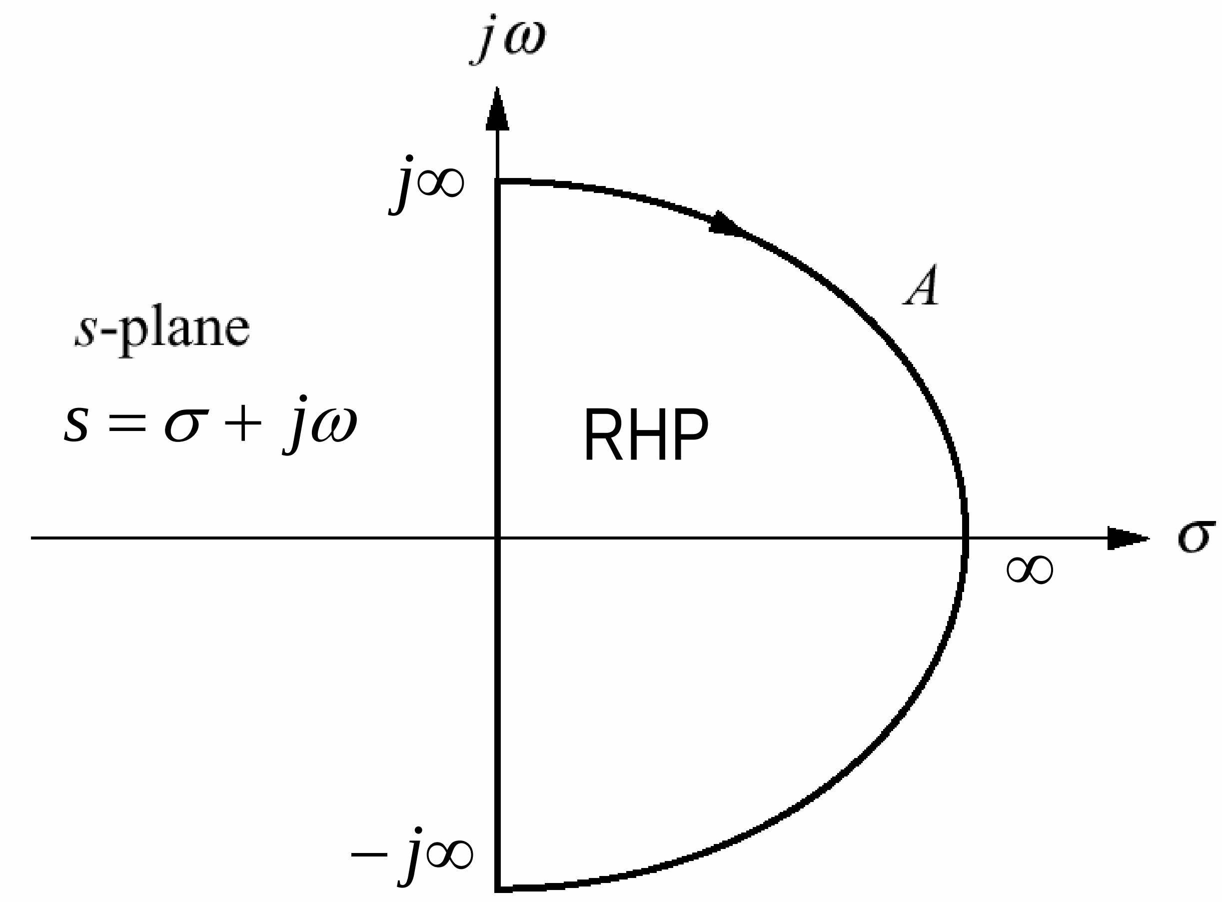
$$G(s) = G_c(s)G_p(s)$$

$H(s)$: Sensor and Filter

The Idea of Mapping



Nyquist Contour



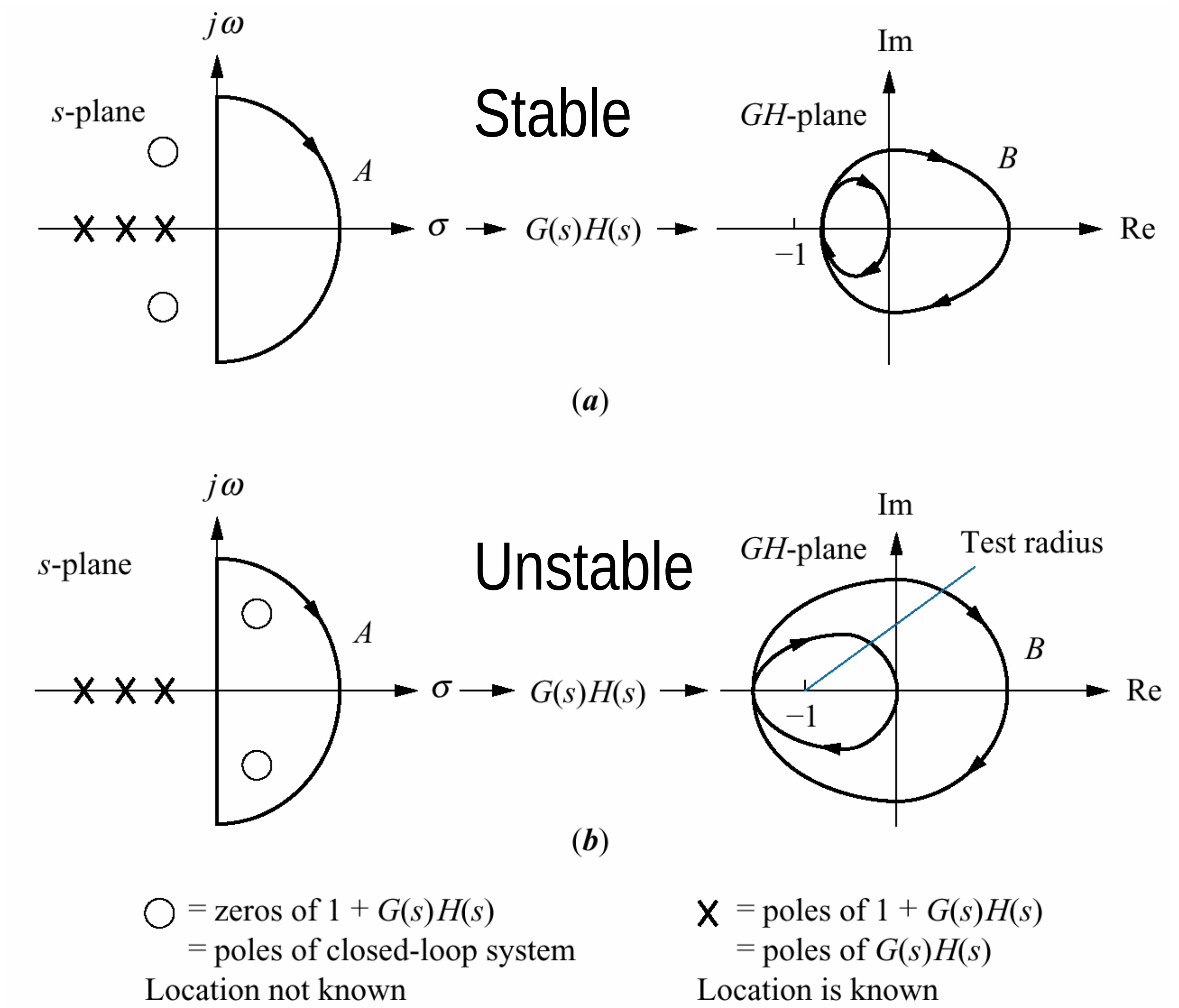
Nyquist Stability Criteria

- Determine stability by inspection
- Assume G(s)H(s) is stable, let s complete the N-countour

  The closed-loop system is stable if G(s)H(s) does not encircle the (-1,0) point

- Basis of Stability Robustness
- Further Reading: unstable G(s)H(s), # of unstable poles

Nyquist Stability Criteria



Stability Robustness

- The (-1,0) point on the GH-plane becomes the focus
- Distance to instability:

  $$G(s)H(s)-(-1)=1+G(s)H(s)$$

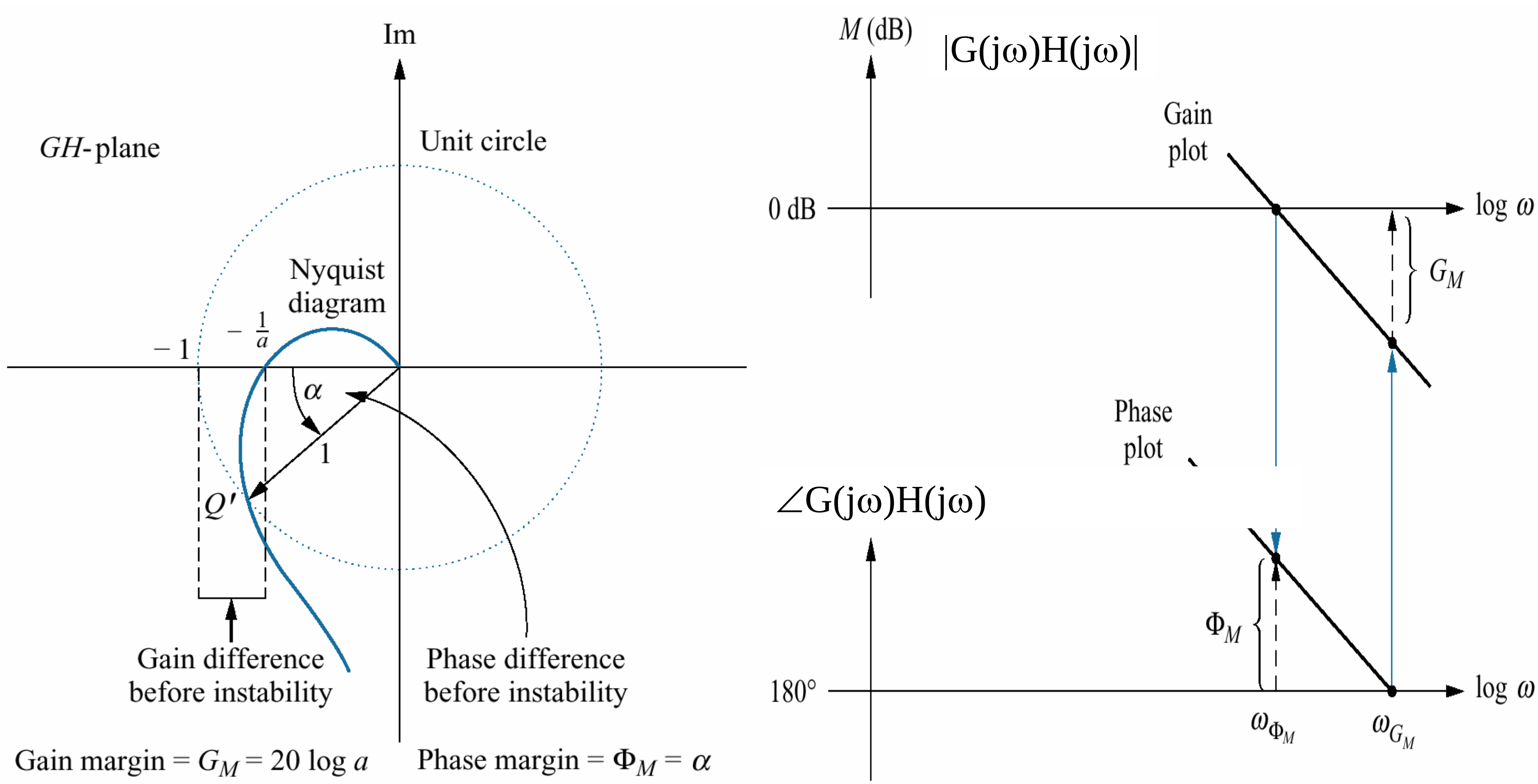
- Robust Stability Condition

  Distance to instability > Dynamic Variations of G(s)H(s)

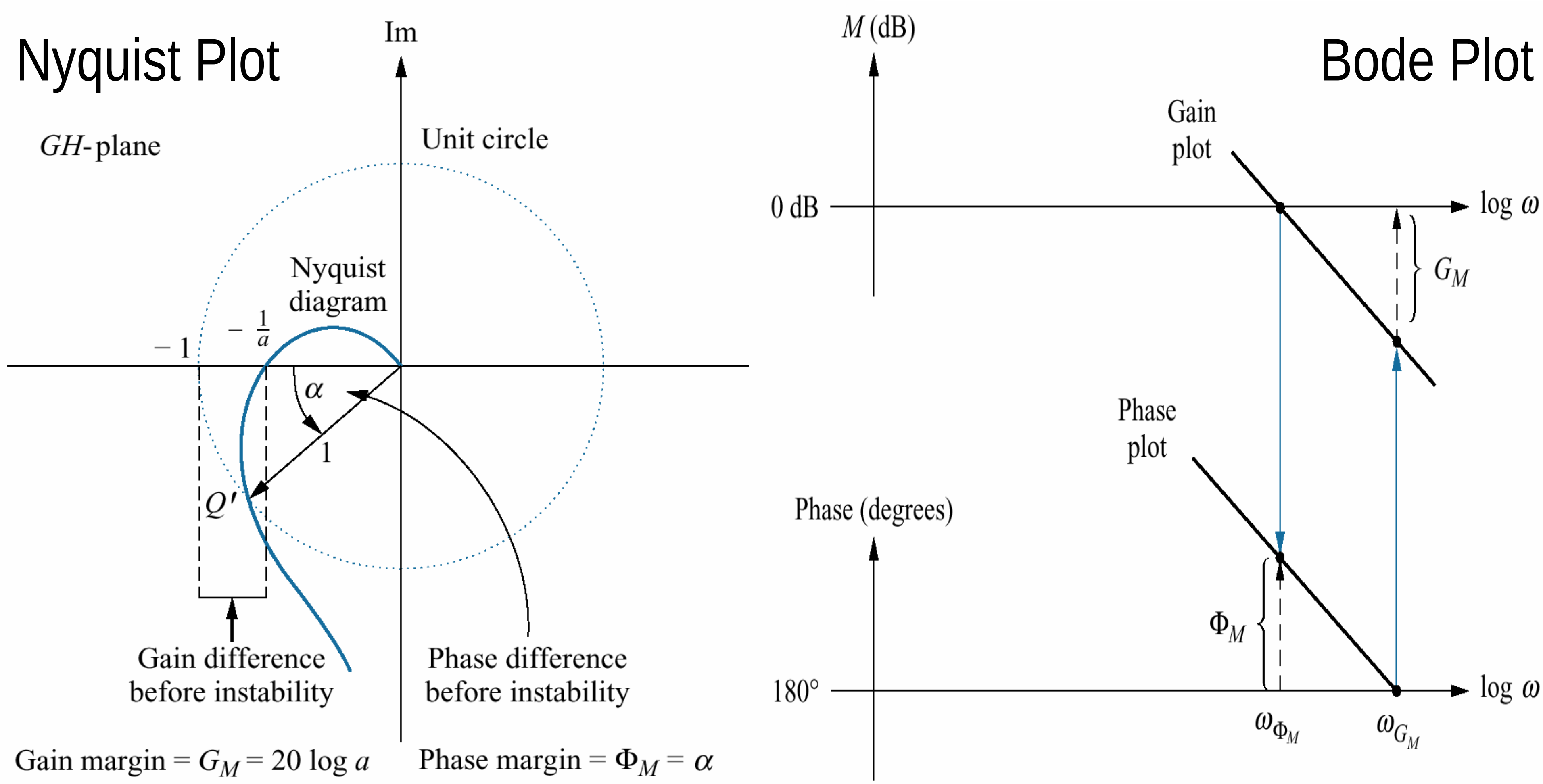
- This is basis of modern robust control theory

Gain and Phase Margins

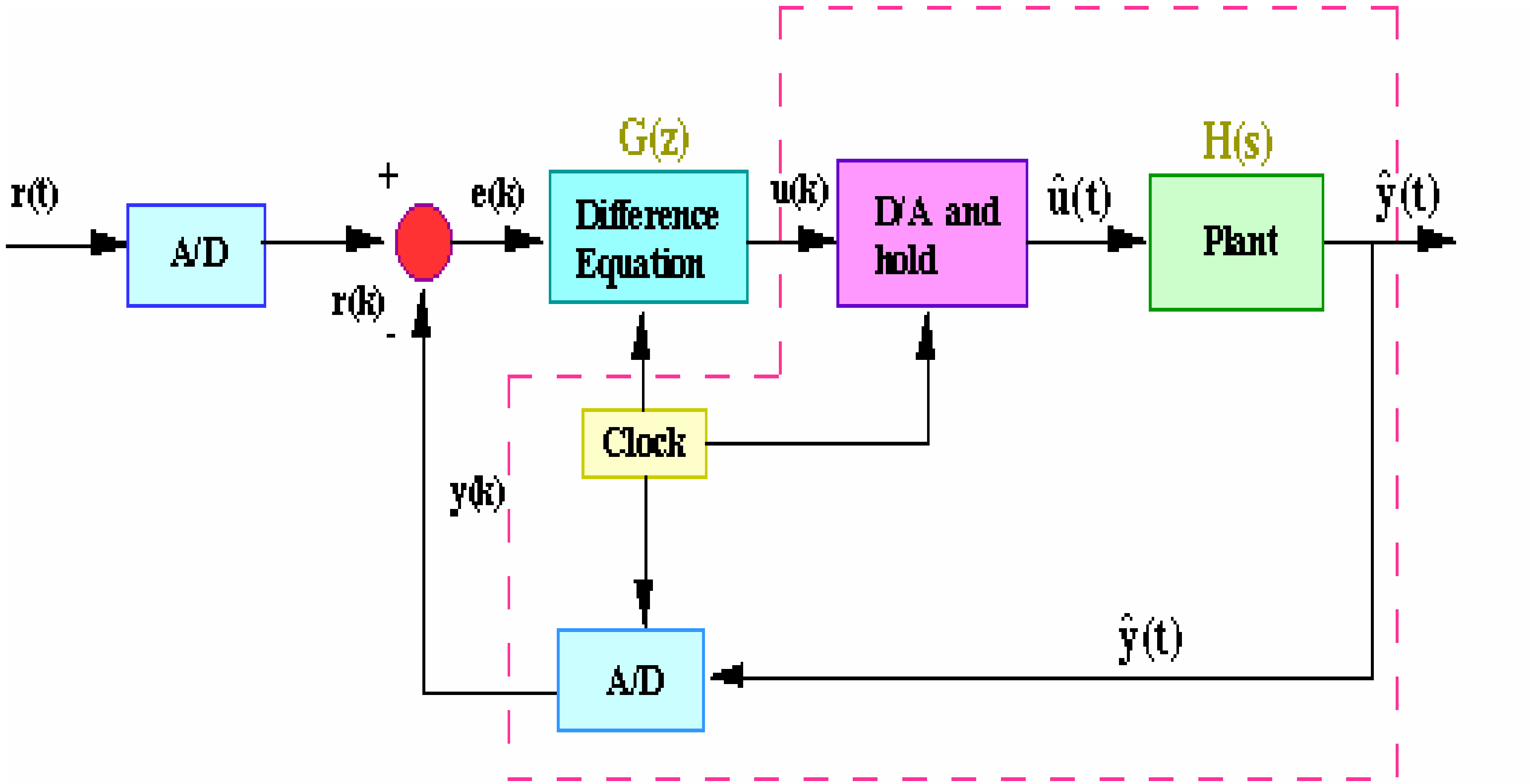
(-1,0) is equivalent of 0db$\angle(-180)°$ point on Bode plot



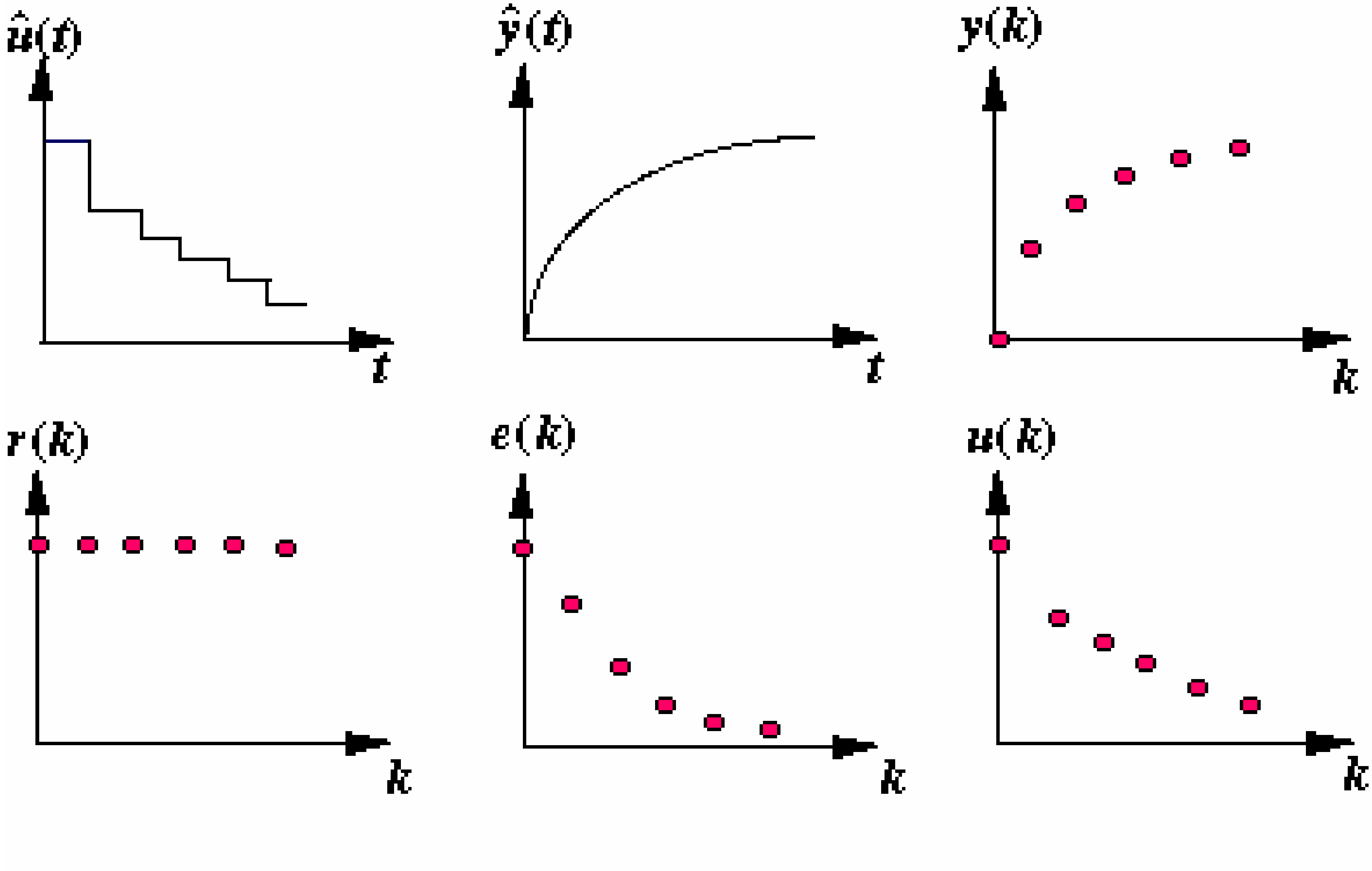
Stability Margins:



Digital Control



Discrete Signals



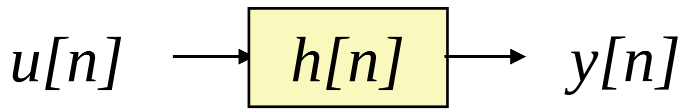
Digital Control Concepts

- Sampling
  - Rate
  - Delay
- ADC and DAC
  - Resolution (quantization levels)
  - Speed
  - Aliasing
- Digital Control Algorithm

  Difference equation

Discrete System Description

- Discrete system



h[n]: impulse response

- Difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k]$$

Discrete Fourier Transform and z-Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Discrete Transfer Function and Frequency Response

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Application of Basic Concepts to Previously Designed Controllers